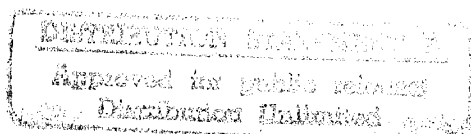


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**NASA CR-2313**

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**NON-LINEAR BEHAVIOR OF  
FIBER COMPOSITE LAMINATES**

*by Zvi Hashin, Debal Bagchi, and B. Walter Rosen*

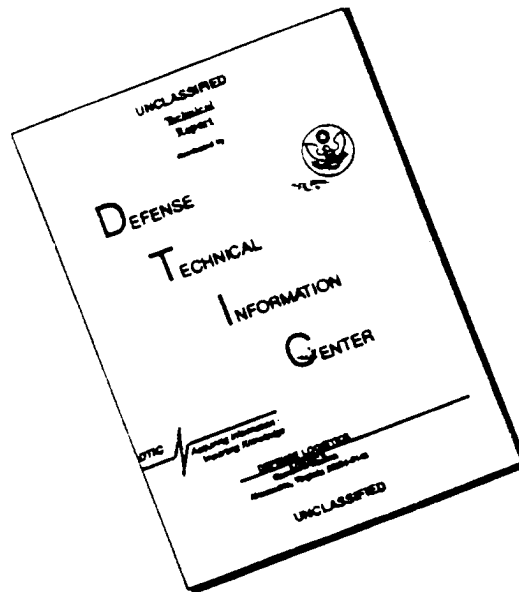
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**NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1974**

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1. Report No. NASA CR-2313		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle NON-LINEAR BEHAVIOR OF FIBER COMPOSITE LAMINATES				5. Report Date APRIL 1974	
				6. Performing Organization Code	
7. Author(s) ZVI HASHIN, DEBAL BAGCHI, AND B. WALTER ROSEN				8. Performing Organization Report No. TFR/7309	
				10. Work Unit No. 501-22-03-02	
9. Performing Organization Name and Address MATERIAL SCIENCES CORPORATION BLUE BELL, PENNSYLVANIA				11. Contract or Grant No. NAS1-11284	
				13. Type of Report and Period Covered CONTRACTOR REPORT	
12. Sponsoring Agency Name and Address NATIONAL AERONAUTICS AND SPACE ADMINISTRATION WASHINGTON, D.C. 20546				14. Sponsoring Agency Code	
15. Supplementary Notes  FINAL REPORT					
16. Abstract  <p>THE NON-LINEAR BEHAVIOR OF FIBER COMPOSITE LAMINATES WHICH RESULTS FROM LAMINA NON-LINEAR CHARACTERISTICS WAS EXAMINED. THE ANALYSIS USES A RAMBERG-OSGOOD REPRESENTATION OF THE LAMINA TRANSVERSE AND SHEAR STRESS STRAIN CURVES IN CONJUNCTION WITH DEFORMATION THEORY TO DESCRIBE THE RESULTANT LAMINATE NON-LINEAR BEHAVIOR.</p> <p>A LAMINATE HAVING AN ARBITRARY NUMBER OF ORIENTED LAYERS AND SUBJECTED TO A GENERAL STATE OF MEMBRANE STRESS WAS TREATED. PARAMETRIC RESULTS AND COMPARISON WITH EXPERIMENTAL DATA AND PRIOR THEORETICAL RESULTS ARE PRESENTED.</p>					
17. Key Words (Suggested by Author(s))			18. Distribution Statement  UNCLASSIFIED - UNLIMITED  CAT. 32		
19. Security Classif. (of this report) UNCLASSIFIED		20. Security Classif. (of this page) UNCLASSIFIED		21. No. of Pages 130	
				22. Price* \$4.75	

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# LIST OF SYMBOLS

$2a, 2b$	- Laminate dimension;
$c$	- Fiber volume fraction;
$C_{ijkl}^*$	- Stiffness matrix;
$E_l$	- Elastic Young's modulus;
$E_A$	- Young's modulus in fiber direction;
$E_T$	- Young's modulus in transverse direction;
$F$	- Function of stress $\sigma_{ij}$ ;
$G_{l,G}$	- Elastic shear modulus of matrix;
$G_A$	- Axial shear modulus;
$G_A^S$	- Effective secant shear modulus of composite;
$ijkl$	- Subscripts ranging from 1 to 3;
$k,m,n$	- Ramberg-Osgood parameters for the matrix;
$I_{1,2,3,4,5}$	- Stress invariants;
$J_2$	- $s_{ij}s_{ij}/2$ ;
$K$	- Number of laminae in a laminate;
$L, \underline{L}$	- Matrices, also quadratic function of stresses;
$M,N$	- Ramberg-Osgood parameters for the composite;
$r,\theta, z$	- Cylindrical coordinate system;
$s_{ij}$	- Stress deviator;
$S_{ij}$	- Compliance matrix;
$S'_{ij}$	- Elastic compliance matrix;
$S''_{ij}$	- Inelastic compliance matrix;
$t$	- Laminate thickness;
$T_i$	- Surface tractions;
$T_{\alpha\beta}$	- Constant edge forces per unit length;
$u_{1,2,3}$	- Displacements in $x_{1,2,3}$ directions, respectively;
$\tilde{u}_i$	- Admissible displacement field;
$U_\epsilon$	- Strain energy;
$U_c$	- Complimentary energy;
$U_p$	- Potential energy;

# SYMBOLS CONTINUED

$V$	- Volume;
$W^\epsilon$	- Strain energy density;
$W^\sigma$	- Complimentary energy density;
$x_{1,2,3}$	- Fixed coordinate directions;
$x_{1,2,3}^{(k)}$	- Local coordinate system for the kth lamina;
$\alpha, \beta$	- Subscripts ranging from 1 to 2;
$\gamma$	- Shear strain;
$\epsilon$	- Strain;
$\bar{\epsilon}_{ij}$	- Average strain tensor;
$\bar{\epsilon}'_{\alpha\beta}$	- Elastic strains;
$\bar{\epsilon}''_{\alpha\beta}$	- Inelastic strains;
$\sigma$	- Stress;
$\sigma_Y$	- Nominal composite yield stress;
$\sigma'$	- Nominal matrix yield stress;
$\underline{\sigma}^\circ$	- Stress $\sigma^\circ_{\alpha\beta}$ at the edges;
$\bar{\sigma}_{ij}$	- Average stress tensor;
$\sigma^\circ_{ij}$	- Applied stress;
$\tau$	- Shear stress;
$\tau_Y$	- Nominal composite yield shear stress;
$\tau_0$	- Average shear stress in composite;
$\tau'_Y$	- Nominal matrix yield shear stress;
$\nu_A$	- Poisson's ratio; and
$\theta_k$	- Reinforcement angle.

NON-LINEAR BEHAVIOR OF  
FIBER COMPOSITE LAMINATES

by Zvi Hashin, Debal Bagchi and B. Walter Rosen  
Materials Sciences Corporation

SUMMARY

The non-linear stress-strain behavior of fiber composite laminates has been analyzed to define the relationship between laminate behavior and the non-linear stress-strain characteristics of unidirectional composites. The resulting analysis has been programmed to yield an efficient computerized design and analysis tool.

The approach utilized herein was to adopt a Ramberg-Osgood representation of the non-linear stress-strain behavior and to utilize deformation theory as an adequate representation of the material nonlinearities. The problem was viewed on two levels. First, the relationship between the constituent properties and the stress-strain response of a unidirectional fiber composite material was studied. For this problem, the primary attention was directed toward axial shear behavior, and an expression was established relating the composite average-stress/average-strain curve to the fiber moduli and the matrix non-linear stress-strain curve. Second level of approach is to treat the interrelationship between the properties of the unidirectional layers and those of the laminate. For this case, the starting point is a non-linear stress-strain curve for transverse stress and for axial shear and a linear stress-strain relation for stress in the fiber direction. The non-linear lamina stress-strain curves can be modeled by proper selection of the Ramberg-Osgood parameters. In the present study, with this as a starting point, an interaction expression was formulated to account for simultaneous application of axial shear and transverse stress.

A laminate having an arbitrary number of oriented layers and subjected to a general state of membrane stress was treated. Parametric results and comparison with experimental data and prior theoretical results are presented.



## 1. INTRODUCTION

A basic requirement for the engineer designing with fiber composite materials is a definition of the stiffness and strength of these materials under a variety of loading conditions, including cases for which experimental materials properties data are not available. For this purpose, it is necessary that he have at his disposal reasonably accurate procedures to predict these mechanical properties. Existing analyses can predict the elastic behavior of a laminated composite quite well when the elastic properties of the unidirectional materials from which it is made are known. However, the situation has been much more complicated and much less satisfactory with regard to the inelastic stiffness and strength of a laminate. The present program was undertaken to develop a computerized analysis of the inelastic behavior of fiber composite laminates which could be used as a design tool. The results of this study and comparisons of these results with experimental data are presented in this report.

It is essential to recognize that the utilization of fiber composite materials in structural design involves the incorporation of material design into the structural design process. This is illustrated clearly by the fact that the gross material properties of a fiber composite laminate change when any change is made in the laminate ply orientations. Even when the designer considers a material formed from a particular combination of fiber and matrix materials, there remains a large number of geometric variables associated with the laminate design. Thus, in the preliminary design phase, experimental material properties data will generally be too limited. In the case of elastic properties, sufficient capability to synthesize the necessary properties exists. This procedure generally starts with the definition of the elastic properties of unidirectional fiber composite materials. These can, of course, be determined experimentally. Also, when such data are not available, they can be estimated using a variety of analytical techniques. These

latter are generally referred to as micromechanics analyses. For example, a set of relatively simple relations for predicting the moduli of unidirectional reinforced composites are presented in [1]. Alternate micromechanics approaches are described in [2] to [4]. A review of these methods is presented in [5]. With these properties available, it is assumed that the individual laminae are homogeneous and anisotropic. A laminate analysis is carried out in a straight forward fashion following methods originally developed for such materials as plywood, and more recently extended to the more general cases associated with fiber composite laminates (e.g., [6] to [8]).

However, contemporary fiber composite materials generally consist of elastic brittle fibers such as glass, boron or graphite in relatively soft matrix materials such as epoxy or aluminum. For these matrix materials it is reasonable to anticipate that at a certain loading state the matrix will begin to exhibit inelastic effects. This results in non-linear relations between structural loads and deformations. These inelastic effects can, of course, be expected to have a significant effect upon failure of the laminate. It is quite clear that adequate definition of these failure conditions are essential to achieve structural designs of high reliability.

In the present study, a non-linear laminate analysis has been developed which can provide realistic assessments of the stresses and strains in the various laminae and of the inelastic stiffnesses of the laminate at any stress level. This information can be used for assessment of such effects as structural stability or structural stress distributions. The stress distributions in the laminae and the laminates can also be utilized for the development of more realistic failure criteria.

Inelastic matrix behavior can be classified broadly as either time dependent or time independent. Time dependent behavior is called viscoelastic if linear and creep if non-linear. Polymeric matrices such as epoxy do exhibit such behavior. In

the case of metallic matrix materials, such as aluminum, time dependent effects are generally negligible unless elevated temperature conditions are considered. The present study is concerned with time independent non-linear matrix behavior which is of significance for both polymeric and metallic matrices. Throughout this paper the expression "inelastic" is used to describe this time independent mechanical behavior. The method of approach to these problems is similar to that of the elastic analysis. Thus, it is necessary to determine, first, the inelastic properties of the unidirectional fiber composite materials. This can be done experimentally or by micromechanics methods. Given this information, a method to determine stresses and strains in an inelastic laminate is then devised. The problem is complicated by the fact that the inelastic stress-strain relations are non-linear.

A limited number of pertinent investigations can be found in the literature. Hill [4] considered, in approximate fashion, a limited aspect of inelastic behavior of a uniaxially reinforced material: the case of stress in fiber direction combined with isotropic transverse stress. Petit and Waddoups [6] devised an incremental method for laminate analysis in which it was assumed that in single laminae there is no interaction of stress components in different directions as far as lamina deformation is concerned. This assumption is restrictive, and also their incremental laminate analysis scheme is unduly complicated. Adams [7] used a finite element technique for numerical analysis of unidirectional materials in the form of periodic fiber arrays under conditions of plane strain. Huang [8] gave an approximate analysis for transverse inelastic behavior for a unidirectional material in plane strain, but it is difficult to assess the validity of the approximations introduced.

A detailed analysis of the inelastic laminate problem has been given by Foye and Baker [9]. Using finite element methods, they computed the inelastic effective properties of unidirectional rectangular and square arrays of elastic fibers

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in inelastic matrix. These properties were then used in an inelastic laminate analysis. The analysis is based on incremental plasticity theory and is, unfortunately, very complicated and requires a great deal of computer time. The results obtained are, however, of great importance for comparison with results predicted by more simplified theories, such as the one which will be given in the present work.

The body of this report is divided into four major sections. In the first, consideration is given to the behavior of unidirectional fiber composite materials. This requires: a definition of the appropriate form of the inelastic stress-strain relations; some consideration of the relationship between composite properties and constituent properties; and a definition of the appropriate form of the interaction between various stress components. The basic objective in this phase of the report was to define appropriate constitutive relations for the individual lamina which can be used in the non-linear laminate analysis. Further, there is a desire to gain some insight into the influence of the particular constituent properties upon the lamina stress-strain relations. In this phase of the study, it is found useful to characterize the unidirectional material with the aid of Ramberg-Osgood stress-strain relations.

In the next section of the report, the analysis of the inelastic behavior of laminates is described. Here, a procedure for incorporating the non-linear constitutive relations into an analysis which defines the state of stress in the individual laminae under an arbitrary set of external loads, is defined. Analyses are developed for the case of symmetric laminates subjected to membrane loading. The equations which are developed uniquely define the desired laminate internal average stress distribution under a given set of membrane loads. Governing equations, however, are non-linear and require numerical solution procedures. An efficient algorithm has been defined which enables computer solution to be achieved for arbitrary

laminates at minimal cost. The solution is obtained by application of the Newton-Raphson method.

In the final section, the computerized analysis which has been developed is applied to series of problems. The first group presents comparisons with various analytical results from the more complex analyses of Ref. [6] and [9]. The second group of numerical results presents comparisons between theoretical results from the present model and available experimental data. The third group of results provides several parametric studies to gain insight into those factors which contribute significantly to the non-linear behavior of fiber composite laminates. Also, computations have been made to provide a preliminary assessment of combined load effects including comparisons with limited experimental data.

Details of the various analytical developments, as well as descriptions of the computer program, are presented in appendices to the report.

The principal result of the present program is a computer program which provides a simple engineering tool which can be used for the parametric study of the influence of material properties upon laminate performance. This laminate analysis capability can be used by the structural designer to define design allowable stresses and to aid in the selection of fiber composite materials for structural applications. A comparison of the present results with the limited amount of available experimental data shows good agreement. There are, however certain cases in which the agreement is not good, particularly as the laminate loading approaches failure. The results of the present analytical method agree well with the results for those problems for which more exact and more complex analytical results exist.

## 2. NON-LINEAR STRESS-STRAIN RELATIONS OF UNIAXIAL FIBER REINFORCED MATERIALS

### 2.1 General Form of Stress-Strain Relations

An effective stress-strain relation of a composite material is defined as a relation between average stress  $\bar{\sigma}_{ij}$  and average strain  $\bar{\epsilon}_{ij}$ . Here and in the following latin indices range over 1, 2, and 3. If the composite is elastic the general effective stress-strain relation takes the form

$$\bar{\sigma}_{ij} = C^*_{ijkl} \bar{\epsilon}_{kl} \quad (2.1.1)$$

where  $C^*_{ijkl}$  are the effective elastic moduli which are material constants and are thus independent of stress or strain. Thus, (2.1.1) is a linear relation between average stress and strain.

If the composite is subject to symmetries the form of (2.1.1) simplifies. For a uniaxial FRM the most important cases of symmetry are transverse isotropy, around fiber direction, and square array (square symmetry). In these cases the stress-strain relations (2.1.1) for transverse isotropy assume the form:

$$\begin{aligned} \bar{\sigma}_{11} &= C^*_{11} \bar{\epsilon}_{11} + C^*_{12} \bar{\epsilon}_{22} + C^*_{12} \bar{\epsilon}_{33} \\ \bar{\sigma}_{22} &= C^*_{12} \bar{\epsilon}_{11} + C^*_{22} \bar{\epsilon}_{22} + C^*_{23} \bar{\epsilon}_{33} \\ \bar{\sigma}_{33} &= C^*_{12} \bar{\epsilon}_{11} + C^*_{23} \bar{\epsilon}_{22} + C^*_{22} \bar{\epsilon}_{33} \\ \bar{\sigma}_{12} &= 2C^*_{44} \bar{\epsilon}_{12} \end{aligned} \quad (2.1.2)$$

$$\bar{\sigma}_{23} = 2C^*_{55} \bar{\epsilon}_{23}$$

$$\bar{\sigma}_{31} = 2C^*_{44} \bar{\epsilon}_{31}$$

and

$$(2.1.3)$$

$$C^*_{55} = (C^*_{22} - C^*_{23})/2$$

In (2.1.2-3) 1 indicates direction and 2, 3 perpendicular directions transverse to 1.

In the event of inelastic matrix and elastic fibers, the situation is much more complicated since the stress-strain

relation are nonlinearity and history dependent. In no case is stress proportional to strain so that superposition of effects is not valid, and in order to determine current strain it is not sufficient to know current stress but it is necessary to know precisely the variation of stress which preceded its current value. Thus, for a material in a known state of combined shear and uniaxial tension, the state of strain is different if: (a) tension is first applied and then the shear, (b) shear is first applied and then the tension- (c) tension and shear are applied simultaneously. For this reason stress-strain relations must be presented in incremental form. That is, strain increment is related to stress and stress increment. This complicates matters enormously. However, it is known that in the case of proportional loading, that is, all stresses at a point grow simultaneously in a fixed ratio to one another, incremental theory can be integrated into the much simpler total or deformation theory for which current strain is completely determined by current stress.

Deformation theories have a wider range of validity than proportional loading. Comparison of numerous detailed solutions carried out both incrementally and by much simpler deformation theory show surprising agreement in many cases, and Budiansky [10] has shown that deformation theory can also be valid for "neighboring" loading paths.

In the present work, we are concerned with composites which are subjected to some external load. If it is supposed that the various external load components grow proportionally, this does not necessarily imply that the components of stress at a typical internal point also grow proportionally. It is, however, felt that the manner of growth of these internal stress components cannot deviate severely from proportional loading if external loading is proportional. Consequently, deformation type stress-strain relations are assumed for the matrix.

This assumption results in considerable simplification. It will be seen that it yields results which are extremely

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close to the ones obtained in [9] on the basis of the much more complicated incremental theory.

It is shown in Appendix A that for elastic fibers and an inelastic matrix described by deformation type theory, the effective stress-strain relations for a transversely isotropic or square symmetric FRM are:

$$\begin{aligned}
 \bar{\epsilon}_{11} &= S_{11} \bar{\sigma}_{11} + S_{12} \bar{\sigma}_{22} + S_{12} \bar{\sigma}_{33} \\
 \bar{\epsilon}_{22} &= S_{12} \bar{\sigma}_{11} + S_{22} \bar{\sigma}_{22} + S_{23} \bar{\sigma}_{33} \\
 \bar{\epsilon}_{33} &= S_{12} \bar{\sigma}_{11} + S_{23} \bar{\sigma}_{22} + S_{22} \bar{\sigma}_{33} \\
 \bar{\epsilon}_{12} &= 2S_{44} \bar{\sigma}_{12} \\
 \bar{\epsilon}_{23} &= 2S_{55} \bar{\sigma}_{23} \\
 \bar{\epsilon}_{13} &= 2S_{44} \bar{\sigma}_{13}
 \end{aligned} \tag{2.1.4}$$

and

$$S_{55} = (S_{22} - S_{23})/2 \tag{2.1.5}$$

The coefficients  $S_{11}$ ,  $S_{12}$ , etc. are the effective inelastic compliances of the material and are functions of the average stresses, or rather of certain invariants of the average stress tensor.

We are here primarily concerned with thin uniaxially reinforced laminae which are in a state of plane stress. Let  $x_1$  denote fiber direction,  $x_2$  direction transverse to fibers in lamina plane, and  $x_3$  direction perpendicular to lamina, Figure 1. Then the plane stress condition is expressed by:

$$\bar{\sigma}_{13} = \bar{\sigma}_{23} = \bar{\sigma}_{33} = 0 \tag{2.1.6}$$

Eqs. (2.1.4) then assume the form:

$$\begin{aligned}
 \bar{\epsilon}_{11} &= S_{11} \bar{\sigma}_{11} + S_{12} \bar{\sigma}_{22} \\
 \bar{\epsilon}_{22} &= S_{12} \bar{\sigma}_{11} + S_{22} \bar{\sigma}_{22} \\
 \bar{\epsilon}_{12} &= 2S_{44} \bar{\sigma}_{12}
 \end{aligned} \tag{2.1.7}$$



Note that  $\bar{\epsilon}_{33}$  does not vanish. It is however of no interest for present purposes.

The inelastic compliances in (2.1.7) are functions of the stresses  $\bar{\sigma}_{11}$ ,  $\bar{\sigma}_{22}$ ,  $\bar{\sigma}_{12}$ .

It is convenient to split the strains in (2.1.7) into elastic strains  $\bar{\epsilon}'_{\alpha\beta}$ , and inelastic strains  $\bar{\epsilon}''_{\alpha\beta}$ . Thus:

$$\bar{\epsilon}_{\alpha\beta} = \bar{\epsilon}'_{\alpha\beta} + \bar{\epsilon}''_{\alpha\beta} \quad (2.1.8)$$

where here and in the following greek indices range over 1, 2. The elastic strains are recovered after unloading of the composite and are related to the stresses by elastic stress-strain relations. Thus:

$$\begin{aligned} \bar{\epsilon}'_{11} &= S'_{11} \bar{\sigma}_{11} + S'_{12} \bar{\sigma}_{22} \\ \bar{\epsilon}'_{22} &= S'_{12} \bar{\sigma}_{11} + S'_{22} \bar{\sigma}_{22} \\ \bar{\epsilon}'_{12} &= 2S'_{44} \bar{\sigma}_{12} \end{aligned} \quad (2.1.9)$$

where

$$\begin{aligned} S'_{11} &= \frac{1}{E_A} & S'_{12} &= -\frac{\nu_A}{E_A} \\ S'_{22} &= \frac{1}{E_T} & S'_{44} &= \frac{1}{4G_A} \end{aligned} \quad (2.1.10)$$

Here  $E_A$  is the effective Young's modulus in fiber direction,  $\nu_A$  - the associated effective Poisson's ration,  $E_T$  - the effective Young's modulus transverse to fibers and  $G_A$  - axial effective shear modulus, related to 1-2 shear.

The inelastic, permanent, strains then have the form:

$$\begin{aligned} \bar{\epsilon}''_{11} &= S''_{11} \bar{\sigma}_{11} + S''_{12} \bar{\sigma}_{22} \\ \bar{\epsilon}''_{22} &= S''_{12} \bar{\sigma}_{11} + S''_{22} \bar{\sigma}_{22} \\ \bar{\epsilon}''_{12} &= 2S''_{44} \bar{\sigma}_{12} \end{aligned} \quad (2.1.11)$$

where

$$S''_{2\beta} = S''_{2\beta} (\bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{12}) \quad (2.1.12)$$

In order to further simplify the stress-strain relations (2.1.11-.12), some specific features of FRM will be taken into account. In such materials, the fibers are by an order of magnitude stiffer than the matrix (for the case of boron and/or graphite fiber in an epoxy matrix the ratio of fiber to matrix Young's modulus can be in excess of 100). The stiffness ratio becomes larger in the inelastic range since the matrix loses stiffness (i.e., flows) while the fibers retain their stiffness. It is, therefore, clear that the stress  $\bar{\sigma}_{11}$  in fiber direction is practically carried by the fibers alone, with insignificant matrix contribution.

On the other hand, the transverse stress  $\bar{\sigma}_{22}$  and the shear stress  $\bar{\sigma}_{12}$  are primarily carried by the matrix with little fiber contribution.

It follows that inelastic behavior of the FRM is produced primarily by  $\bar{\sigma}_{22}$  and  $\bar{\sigma}_{12}$  while inelastic behavior for  $\bar{\sigma}_{11}$  load can be neglected.

The foregoing comments are summarized into two basic assumptions:

- (a) the inelastic strains  $\bar{\epsilon}_{22}''$  and  $\bar{\epsilon}_{12}''$  are not functions of  $\bar{\sigma}_{11}$
- (b) the inelastic strain  $\bar{\epsilon}_{11}''$  always vanishes.

On the basis of these assumptions, the stress-strain relations (2.1.11-.12) simplify to:

$$\begin{aligned}\bar{\epsilon}_{11}'' &= 0 \\ \bar{\epsilon}_{22}'' &= S_{22}'' (\bar{\sigma}_{22}, \bar{\sigma}_{12}) \bar{\sigma}_{22} \\ \bar{\epsilon}_{12}'' &= 2S_{44}'' (\bar{\sigma}_{22}, \bar{\sigma}_{12}) \bar{\sigma}_{12}\end{aligned}\tag{2.1.13}$$

## 2.2 Plane Stress-Strain Relations in Ramberg-Osgood Form

A convenient representation of non-linear one dimensional stress-strain relations has been given by Ramberg and Osgood [11]. For uniaxial stress, for example:

$$\epsilon = \frac{\sigma}{E_1} \left[ 1 + k \left( \frac{\sigma}{\sigma'} \right)^{m-1} \right] \quad (2.2.1)$$

where  $E_1$  represents the elastic Young's modulus, and  $k$ ,  $\sigma'$ , and  $m$  are three parameters to be obtained by curve fitting. The parameter  $\sigma'$  is sometimes called nominal yield stress. Equation (2.2.1) represents a family of curves with initial slope  $E_1$ , and monotonically decreasing slope with increasing  $\sigma$ . The curves flatten out with increasing  $m$  (Fig. 2). Without loss of generality (2.2.1) can be written in the form:

$$\epsilon = \frac{\sigma}{E_1} \left[ 1 + \left( \frac{\sigma}{\sigma_y} \right)^{m-1} \right] \quad (2.2.2)$$

which will be used from now on. Similarly, a stress-strain curve in shear can be represented in the form:

$$\gamma = \frac{\tau}{G_1} \left[ 1 + \left( \frac{\tau}{\tau_y} \right)^{n-1} \right] \quad (2.2.3)$$

where  $G_1$  is the elastic shear modulus.

It should be emphasized that (2.2.2-3) are valid only for one dimensional cases. The question of the generalization to general states of stress and strain has no unique answer. One common used form is isotropic  $J_2$  deformation theory [12].

Next, we consider the case of effective or macroscopic stress-strain relations for the special case of a uniaxially reinforced material in which the matrix is non-linear, with stress-strain relations in Ramberg-Osgood form.

Consider, for example, the case of uniaxial average stress  $\bar{\sigma}_{22}$  in direction transverse to fibers, all other average stresses vanish. It then follows from (2.1.7) that:

$$\bar{\epsilon}_{22} = S_{22} (\bar{\sigma}_{22}) \bar{\sigma}_{22} \quad (2.2.4)$$

Similarly, if the only nonvanishing average stress is  $\bar{\sigma}_{12}$ , the shear stress-strain relation of the composite is:

$$\bar{\epsilon}_{12} = 2S_{44} (\bar{\sigma}_{12})^{\bar{\sigma}_{12}} \quad (2.2.5)$$

Evidently the inelastic effective compliances  $S_{22}$  and  $S_{44}$  are functions of the parameters of the inelastic Ramberg-Osgood stress-strain relations of the matrix, of the elastic properties of the fibers and of the internal geometry of the composite. Actual prediction is a very difficult problem. Such problems will be considered in limited fashion in the next paragraph.

Just as matrix stress-strain relations are represented in Ramberg-Osgood form, the same type of curve fitting can also be applied for the effective stress-strain relation of the composite. Thus (2.2.2-.3) are written in the form:

$$\bar{\epsilon}_{22} = \frac{\bar{\sigma}_{22}}{E_T} \left[ 1 + \left( \frac{\bar{\sigma}_{22}}{\sigma_Y} \right)^{M-1} \right] \quad (a)$$

$$\bar{\epsilon}_{12} = \frac{\bar{\sigma}_{12}}{2G_A} \left[ 1 + \left( \frac{\bar{\sigma}_{12}}{\tau_Y} \right)^{N-1} \right] \quad (b) \quad (2.2.6)$$

Where  $E_T$  is the effective transverse elastic Young's modulus  $G_A$  - effective axial elastic shear modulus and  $\sigma_Y$ ,  $\tau_Y$ ,  $M$  and  $N$  are curve fitting parameters which are in general quite different from the corresponding Ramberg-Osgood matrix parameters.

A question of fundamental and of practical importance is the form of the stress-strain relations for the case of plane stress, taking into account interaction among the various stress components. It should be noted in this respect that (2.2.6) are special stress-strain relations when  $\bar{\sigma}_{22}$  or  $\bar{\sigma}_{12}$  act only by themselves.

It is recalled that equations (2.1.13) represent the inelastic parts of the strains for plane stress-strain relations for FRM with stiff fibers. It is shown in Appendix B

that the Ramberg-Osgood form of such plane stress-strain relations is as follows:

$$\begin{aligned}\bar{\epsilon}_{11}'' &= 0 \\ \bar{\epsilon}_{22}'' &= \frac{\bar{\sigma}_{22}}{E_T} \left[ \left( \frac{\bar{\sigma}_{22}}{\sigma_Y} \right)^2 + \left( \frac{\bar{\sigma}_{12}}{\tau_Y} \right)^2 \right]^{\frac{M-1}{2}} \\ \bar{\epsilon}_{12}'' &= \frac{\bar{\sigma}_{12}}{2G_A} \left[ \left( \frac{\bar{\sigma}_{22}}{\sigma_Y} \right)^2 + \left( \frac{\bar{\sigma}_{12}}{\tau_Y} \right)^2 \right]^{\frac{N-1}{2}}\end{aligned}\quad (2.2.7)$$

The parameters  $E_T$ ,  $G_A$ ,  $\sigma_Y$ ,  $\tau_Y$ ,  $M$ ,  $N$  in (2.2.7) are those of the one dimensional stress-strain relations (2.2.6) which may be regarded as experimentally (or perhaps theoretically) known.

The inelastic parts of the strains are given by (2.1.9-.10), and the total strains are then given by adding equations (2.2.7) and (2.1.9).

Equations (2.2.7) have been compared with computed numerical results given in [9]. Reasonable agreement was obtained. Comparisons for the interaction cases of transverse stress,  $\bar{\sigma}_{22}$ , versus transverse strain,  $\bar{\epsilon}_{22}$ , in the presence of axial shear stress,  $\bar{\sigma}_{12}$ , and axial shear stress,  $\bar{\sigma}_{12}$ , versus axial shear strain,  $\bar{\gamma}_{12}$ , are shown in Figures 3 and 4 respectively (in both cases  $\bar{\sigma}_{22}/\bar{\sigma}_{12} = 8/3$ ). It is seen that the agreement is fair for transverse stress-strain relations (Fig. 3) and very good for the shear stress-strain relations (Fig. 4).

Figures 3 and 4 also show the stress-strain relations obtained from Eqs. (2.2.7) for one dimensional transverse tension  $\bar{\sigma}_{22}$ , and axial shear,  $\bar{\sigma}_{12}$ , respectively.

### 2.3 Axial Shear Stress-Strain Relation

This paragraph is concerned with the problem of prediction of a one dimensional effective axial shear stress-strain relation of a uniaxial FRM in terms of matrix and fiber properties and the internal geometry of the composite.

The main reason for concentrating on the axial shear problem is that the inelastic effect is predominant in axial shear for which significant nonlinearity of the stress-strain response is obtained (e.g., Figure 4). The effect in fiber direction is practically non-existent as has indeed been assumed above, and is relatively small in transverse stress which is shown by the small curvature of the stress-strain relation in this case (e.g., Figure 3).

On the basis of all this, it can indeed be assumed as first approximation that the nonlinearity of the uniaxial FRM is limited to axial shear alone.

Consider a uniaxially reinforced lamina which is subjected to pure axial shear, Figure 5, on its surface. The boundary conditions are:

$$\begin{aligned} x_3 &= \pm t/2 & \sigma_{31} &= \sigma_{32} = \sigma_{33} = 0 \\ x_2 &= \pm b & \sigma_{12} &= \tau_0 & \sigma_{22} &= \sigma_{23} = 0 \\ x_1 &= \pm a & \sigma_{12} &= \tau_0 & \sigma_{11} &= \sigma_{13} = 0 \end{aligned} \quad (2.3.1)$$

It may be shown that under such load the only nonvanishing average stress in the composite is:

$$\bar{\sigma}_{12} = \tau_0 \quad (2.3.2)$$

It would seem at first that, given the complexity of the internal geometry of the composite, the state of stress at any interior matrix or fiber point is generally three dimensional. Surprisingly enough, however, this is not so and the only non-vanishing stress components in the interior of the composites

are the shear stresses  $\sigma_{12}$  and  $\sigma_{13}$ , which are moreover functions of  $x_2$  and  $x_3$  only. Thus, the interior state of stress is:

$$\begin{aligned}\sigma_{12} &= \sigma_{12}(x_2, x_3) \\ \sigma_{13} &= \sigma_{13}(x_2, x_3) \\ \sigma_{11} &= \sigma_{22} = \sigma_{33} = \sigma_{23} = 0\end{aligned}\tag{2.3.3}$$

The validity of equations (2.3.3) for the case of an elastic composite has been proved in [5]. Their validity for the present much more general inelastic case will be shown elsewhere.

The effective stress-strain relation of the composite in axial shear is defined by:

$$\begin{aligned}\bar{\epsilon}_{12} &= \frac{\sigma_{12}}{2G_A^S} = \frac{\tau_o}{2G_A^S} \\ G_A^S &= G_A^S(\bar{\sigma}_{12}) = G_A^S(\tau_o)\end{aligned}\tag{2.3.4}$$

where  $G_A^S$  is the effective secant shear modulus of the material. The nonlinearity of the stress-strain relation is expressed by the fact that  $G_A^S$  function of the applied stress.

It is seen that in order to determine  $G_A^S$  it is necessary to compute the average shear strain  $\bar{\epsilon}_{12}$  for given applied shear stress. This is a formidable problem even with the simplification (2.3.3) and we shall content ourselves with a brief outline of its formulation. To simplify matters, the fibers shall be assumed to be ideally rigid relative to the matrix. This is a very accurate assumption for the case of Boron and Graphite Fibers. There is no difficulty to extend the formulation to the case of non-rigid elastic fibers.

In view of (2.3.3) the problem is two dimensional and need only be considered in a typical  $x_2, x_3$  section. In the matrix domain:

$$\frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = 0 \quad (2.3.5a)$$

$$\epsilon_{12} = \frac{\sigma_{12}}{2G} \left[ 1 + \left( \frac{\tau}{\tau_Y} \right)^{n-1} \right] \quad (2.3.5b)$$

$$\epsilon_{13} = \frac{\sigma_{13}}{2G} \left[ 1 + \left( \frac{\tau}{\tau_Y} \right)^{n-1} \right] \quad (2.3.5c)$$

$$\tau = \sqrt{\sigma_{12}^2 + \sigma_{13}^2} \quad (2.3.6)$$

$$\epsilon_{12} = \frac{1}{2} \frac{\partial u_1}{\partial x_2} \quad (2.3.7a)$$

$$\epsilon_{13} = \frac{1}{2} \frac{\partial u_1}{\partial x_3} \quad (2.3.7b)$$

$$u_1 = u_1(x_2, x_3) \quad (2.3.8)$$

and,  $u_1 = 0$  at fiber/matrix interface.

Here equ. (2.3.5) is the only surviving equilibrium equation, (2.3.6) are Ramberg-Osgood stress-strain relations for isotropic  $J_2$  theory (2.3.7) are usual strain-displacement relations in which  $u_2$  and  $u_3$  do not enter since it may be shown that they are not functions of  $x_1$  and (2.3.8) expresses the ideal rigidity of the fibers.

Eqs. (2.3.5-.8) must be solved subject to boundary condition (2.3.1). If this is done the strain  $\epsilon_{12}$  is known everywhere and can be averaged to obtain  $G_A^S$  from (2.3.4).

The problem is exceedingly difficult because of the non-linearity introduced by the stress-strain relations (2.3.6). There is very little hope to solve it analytically for any kind of fiber geometry. It should therefore be handled by numerical methods for fiber arrangements and fiber shapes of engineering interest.

Another way to approach an analytically intractable problem such as the present one is by variational techniques.



In this fashion, approximations or bounds for quantities of interest are obtained by methods which are much simpler than bonafide solution of the problem. Such variational methods have been extensively used for determination of effective elastic moduli of FRM (e.g., [5]).

In the course of the present work, it has been found that variational methods can also be used for inelastic problems such as the present one to obtain bounds on effective secant moduli. The main ingredients of the method are:

- (a) Construction of an extremum principle in terms of an energy integral such that the true energy is the minimum of the integral.
- (b) Expression of the true energy in terms of effective secant modulus.
- (c) Establishment of admissible fields to obtain a value of the energy integral which is larger than the true energy, thus obtaining a bound for  $G_A^S$ .

The work involves complicated developments and derivations which are given in Appendix C. Here only the end result for a lower bound on  $G_A^S$  will be given for a special geometry of FRM which is known as composite cylinder assemblage. This geometry has been described in detail in [1, 5] and consists of an assemblage of composite cylinders of variable sizes which are joined together so as to fill the whole volume of the composite. In order to fill the whole volume, composite cylinders vary from finite to infinitesimal size. This geometry has been used to advantage for elastic FRM to obtain simple expressions for effective elastic moduli which are well verified by experiment [1, 5]. In the present case only a lower bound on  $G_A^S$  has been obtained for the case in which the exponent  $n$  in matrix stress-strain relations is  $n=3$ .

It has been found that with this exponent and proper choice of  $\tau_y$ , epoxy shear stress-strain relations can be well described. The result for the lower bound is:

$$G_A^S \geq G_{A(-)}^S = \frac{G \frac{1+c}{1-c}}{1 + \left(\frac{\tau_0}{\tau_Y'}\right)^2 \frac{3+13c+c^2+c^3}{3(1+c)^3}} \quad (2.3.9)$$

where

$c$  - volume fraction of fibers

$G$  - elastic (initial) matrix shear modulus

$\tau_Y'$  - Ramberg-Osgood matrix stress parameter, and

$\tau_0$  - applied shear stress.

It follows from (2.3.4) that:

$$\bar{\epsilon}_{12} \leq \frac{\tau_0}{2G_{A(-)}^S} \quad (2.3.10)$$

In other words, with the lower bound on  $G_A^S$  an upper bound on  $\bar{\epsilon}_{12}$  variation with  $\tau_0$  is obtained.

If (2.3.10) is explicitly written in terms of (2.3.9) it assumes the form:

$$\bar{\epsilon}_{12} \leq \frac{\tau_0}{2G \frac{1+c}{1-c}} \left[ 1 + \left(\frac{\tau_0}{\tau_Y'}\right)^2 \frac{3+13c+c^2+c^3}{3(1+c)^3} \right] \quad (2.3.11)$$

Recalling that for the composite cylinder assemblage with rigid fibers the axial elastic shear modulus  $G_A$  is given in [1, 5] as:

$$G_A = G \frac{1+c}{1-c} \quad (2.3.12)$$

and comparing (2.3.11) with (2.2.6) with choice of exponent  $N=3$  (which is the same as matrix exponent), it is seen that:

$$\tau_Y^2 \geq \tau_Y'^2 \frac{3(1+c)^3}{3+13c+c^2+c^3} \quad (2.3.13)$$

The prediction of (2.3.11) has been compared with numerical results obtained in [9]. Figure 6 shows the variation of the right side of (2.3.11) in comparison with the results obtained in [9] for a fiber volume fraction,  $c=0.5$ . Since results of [9] were for boron fibers in epoxy matrix, the rigid fiber approximation is accurately valid. It is seen that the results are reasonably close. It should be noted that the geometry of [9] is a rectangular fiber array which is quite different from the composite cylinder assemblage geometry.

The results defined by (2.3.12) and (2.3.13) used in equation (2.2.6) yield the result plotted in non-dimensional form in Fig. 7. The shear strains are normalized with respect to the matrix elastic strain,  $\gamma_{ye}$ , at the yield stress,  $\tau_Y$ :

$$\gamma_{ye} = \frac{\tau_Y}{G} \quad (2.3.14)$$

It is natural to also consider the establishment of an upper bound on  $G_A^S$ . Unfortunately, however, this is a matter of formidable difficulty for the reason that inversion of (2.3.6) to express stresses in terms of strains leads very complicated expressions. Further discussion of this difficulty is given in Appendix C.

### 3. ANALYSIS OF NON-LINEAR LAMINATES

#### 3.1 Formulation

The general problem to be investigated in the present chapter is as follows: given the inelastic stress-strain relations of uniaxially reinforced laminae determined theoretically or experimentally, and a laminate composed of such laminae and loaded on its edges by uniformly distributed loads in the plate of the laminate:

- (a) What are the stresses in the various laminae?
- (b) What is the macroscopic strain response of the laminate to the loads?

This problem has been extensively investigated for elastic laminates, and the results obtained will serve as important guidelines for the present much more complicated problem. It is therefore very helpful to first briefly review the theory of elastic laminates.

Let the laminate be referred to a fixed system of coordinates  $x_1, x_2, x_3$  as shown in Figure 8. This will henceforth be referred to as the laminate coordinate system.

Any lamina,  $k$ th say, in the laminate will be referred to its material system of coordinates  $x_1^{(k)}, x_2^{(k)}, x_3$  where  $x_1^{(k)}$  is in fiber direction,  $x_2^{(k)}$  perpendicular to fiber direction and  $x_3$  is the same as the laminate  $x_3$ , Figure 8. The reinforcement angle  $\theta_k$  is defined by:

$$\theta = \angle (x_1, x_1^{(k)}) = \angle (x_2, x_2^{(k)}) \quad (3.1.1)$$

Let it be assumed that the laminae are in states of plane stress. It will be later explained under what conditions this is true. Then the stress-strain relations of a single lamina referred to its material coordinate system are written in the forms:

$$\epsilon_{\alpha\beta}^{(k)} = S_{\alpha\beta\gamma\delta}^{(k)} \sigma_{\gamma\delta}^{(k)} \quad (a) \quad (3.1.2)$$

$$\underline{\epsilon}^{(k)} = \underline{S}^{(k)} \underline{\sigma}^{(k)} \quad (b) \quad 21.$$

where (3.1.2a) is in tensor notation with range of subscripts 1, 2 and (3.1.2b) is in matrix notation. It should be noted that (3.1.2) represent the stress-strain relations (2.1.9 - .10), i.e.,

$$\begin{aligned}\epsilon_{11}^{(k)} &= \frac{\sigma_{11}^{(k)}}{E_A^{(k)}} - \frac{\nu_A^{(k)}}{E_A^{(k)}} \sigma_{22}^{(k)} \\ \epsilon_{22}^{(k)} &= - \frac{\nu_A^{(k)}}{E_A^{(k)}} \sigma_{11}^{(k)} + \frac{\sigma_{22}^{(k)}}{E_T^{(k)}} \\ \epsilon_{12}^{(k)} &= \frac{\sigma_{12}^{(k)}}{2G_A^{(k)}}\end{aligned}\quad (3.1.3)$$

Let a laminate of rectangular form, Figure 8, be loaded by a uniform edge stress:

$$\begin{aligned}\sigma_{11}(\pm a, x_2) &= \sigma_{11}^0 \\ \sigma_{12}(\pm a, x_2) &= \sigma_{12}^0 \\ \sigma_{12}(x_1, \pm b) &= \sigma_{12}^0 \\ \sigma_{22}(x_1, \pm b) &= \sigma_{22}^0\end{aligned}\quad (3.1.4)$$

The elasticity solution of the laminate must satisfy the following requirements:

- (a) Equilibrium of stresses,
- (b) Traction continuity at laminae interfaces,
- (c) Boundary conditions (3.1.4), and
- (d) Displacement continuity at laminae interfaces.

It is assumed that the stresses in any lamina are constant, but different in the different laminae. The condition (a) is satisfied within any lamina. Since the assumed lamina stresses are plane there are no traction components on laminae interfaces. Therefore (b) is satisfied.

The boundary conditions (3.1.4) cannot be strictly satisfied in each lamina but only in an average sense. To do this lamina stresses  $\sigma_{\alpha\beta}^{(k)}$  referred to lamina material coordinates

are transformed to laminate axes. The stresses in the kth lamina referred to laminate axes are denoted  $^{(k)}\sigma_{\alpha\beta}$ . The transformation is given by:

$$\begin{aligned}^{(k)}\sigma_{11} &= \sigma_{11}^{(k)} \cos^2 \theta_k + \sigma_{22}^{(k)} \sin^2 \theta_k - 2\sigma_{12}^{(k)} \cos \theta_k \sin \theta_k \\^{(k)}\sigma_{22} &= \sigma_{22}^{(k)} \sin^2 \theta_k + \sigma_{11}^{(k)} \cos^2 \theta_k + 2\sigma_{12}^{(k)} \cos \theta_k \sin \theta_k \\^{(k)}\sigma_{12} &= (\sigma_{11}^{(k)} - \sigma_{22}^{(k)}) \sin \theta_k \cos \theta_k + \sigma_{12}^{(k)} (\cos^2 \theta_k - \sin^2 \theta_k)\end{aligned} \quad (3.1.5)$$

or in matrix notation:

$$^{(k)}\underline{\sigma} = \underline{\theta}^{(k)} \underline{\sigma}^{(k)} \quad (3.1.6)$$

Let the edges of the laminate be loaded by constant forces per unit length  $T_{11}$ ,  $T_{22}$ ,  $T_{12}$  and define the stresses (3.1.4) as edge averages over the laminate thickness  $h$ :

$$\begin{aligned}\sigma_{11}^{\circ} &= T_{11}/h \\ \sigma_{22}^{\circ} &= T_{22}/h \\ \sigma_{12}^{\circ} &= T_{12}/h\end{aligned} \quad (3.1.7)$$

Equilibrium requires that:

$$\begin{aligned}\sum_{k=1}^K \sigma_{11}^{(k)} &= \sigma_{11}^{\circ} \\ \sum_{k=1}^K \sigma_{22}^{(k)} &= \sigma_{22}^{\circ} \\ \sum_{k=1}^K \sigma_{12}^{(k)} &= \sigma_{12}^{\circ}\end{aligned} \quad (3.1.8)$$

where  $K$  is the number of laminae. Written in terms of stresses  $^{(k)}\sigma_{\alpha\beta}$  using (3.1.6), we have:

$$\sum_{k=1}^K \underline{\theta}^{(k)} \underline{\sigma}^{(k)} = \underline{\sigma}^{\circ} \quad (3.1.9)$$

where  $\underline{\sigma}^{\circ}$  denotes the stresses  $\sigma_{\alpha\beta}^{\circ}$  at the edges.

Replacement of the boundary conditions (3.1.4) by (3.1.6) is an approximation of Saint Venant type. Thus, there must be expected edge perturbations (among them interlaminar shear) on the stresses predicted by laminate theory.

Equations (3.1.8) are three equations for the 3K stresses  $\sigma_{\alpha\beta}^{(1)}, \sigma_{\alpha\beta}^{(2)} \dots \sigma_{\alpha\beta}^{(K)}$  in the laminae. There are needed an additional  $3(K-1)$  equations which are provided by displacement continuity at lamina interfaces, requirement (d).

Since the stresses in each laminae are by hypothesis uniform, so are the strains. Therefore, displacement continuity is ensured if the lamina strains in adjacent laminae, referred to laminae coordinate system are the same. Thus:

$$\begin{aligned} {}^{(k)}\epsilon_{11} &= {}^{(k+1)}\epsilon_{11} \\ {}^{(k)}\epsilon_{22} &= {}^{(k+1)}\epsilon_{22} \\ {}^{(k)}\epsilon_{12} &= {}^{(k+1)}\epsilon_{12} \end{aligned} \quad k=1,2,\dots,k \quad (3.1.10)$$

Equations (3.1.10) are the additional required  $3(K-1)$  equations. They will be written in terms of laminae stresses  $\sigma_{\alpha\beta}^{(k)}$  referred to laminae material axes. To do this it is noted that:

$${}^{(k)}\underline{\epsilon} = \underline{\theta}^{(k)} \underline{\epsilon}^{(k)}$$

which is just a transformation of (3.1.6). From (3.1.2b):

$${}^{(k)}\underline{\epsilon} = \underline{\theta}^{(k)} \underline{S}^{(k)} \underline{\sigma}^{(k)} \quad (3.1.11)$$

and inserting the last result in (3.1.10):

$$\underline{\theta}^{(k)} \underline{S}^{(k)} \underline{\sigma}^{(k)} = \underline{\theta}^{(k+1)} \underline{S}^{(k+1)} \underline{\sigma}^{(k+1)} \quad k=1,2,\dots,k \quad (3.1.12)$$

Equations (3.1.9) and (3.1.12) are 3K linear equations for the 3K stresses in an elastic laminate, with K layers.

It should be carefully noted that the analysis given above is based on plane stress conditions in individual laminae. This

is a valid assumption if:

- (a) The loads on the laminate are statically equivalent to in-plane forces (membrane forces) and produce neither bending nor twisting moments, and
- (b) The laminate has a certain stacking sequence of laminae which defines a so called balanced or symmetric laminate.

This stacking sequence is an arrangement in which the laminate has a middle plane of geometrical and of material symmetry. The laminae are arranged in pairs with respect to the plane of symmetry. The laminae of such pair have equal thicknesses, same distances from middle plane, and are of the same material with same angles of reinforcement.

In a non-symmetric laminate application of membrane forces will in general produce bending and twisting of laminae and thus a plane state of stress will not be realized. The symmetric laminate is, however, sufficiently versatile to cover most cases of practical interest.

Let it now be assumed that the laminate is inelastic but still fulfills the conditions of symmetry and pure membrane loading. In this case the only equations which necessarily change in the preceding development are the stress-strain relations of the laminae, (3.1.2), which must be replaced by inelastic laminae stress-strain relations are given by (2.1.7) where the compliances are now functions of the stresses. These compliances now replace the elastic compliances in (3.1.2) which thus become non-linear.

It is convenient for later purposes to rewrite (3.1.2) in the inelastic case in different form. To do this the strains  $\epsilon_{\alpha\beta}^{(k)}$  are first split into elastic strains (2.1.9) and inelastic strains (2.1.11). Preceding to (3.1.12) this equation assumes the form:

$$\begin{aligned} \underline{\epsilon}^{(k+1)} \underline{S}^{1(k+1)} \underline{\sigma}^{(k+1)} - \underline{\epsilon}^{(k)} \underline{S}^{1(k)} \underline{\sigma}^{(k)} = \\ - \underline{\epsilon}^{(k+1)} \underline{S}^{11(k+1)} \underline{\sigma}^{(k+1)} + \underline{\epsilon}^{(k)} \underline{S}^{11(k)} \underline{\sigma}^{(k)} \end{aligned} \quad (3.1.13)$$

$$k = 1, 2, \dots, k$$



where

$\underline{s}^{1(k)}$  - elastic compliance matrix of kth layer

$\underline{s}^{11(k)}$  - inelastic part of compliance matrix of kth layer

$$\underline{s}^{(k)} = \underline{s}^{1(k)} + \underline{s}^{11(k)} \quad (3.1.14)$$

Equations (3.1.13) are now written out in component form with notation (2.1.10), (2.1.12) for compliances:

$$\begin{aligned} & \sigma_{11}^{(k+1)} (s_{11}^{1(k+1)} \cos^2 \theta_{k+1} + s_{12}^{1(k+1)} \sin^2 \theta_{k+1}) \\ & + \sigma_{22}^{(k+1)} (s_{12}^{1(k+1)} \cos^2 \theta_{k+1} + s_{22}^{1(k+1)} \sin^2 \theta_{k+1}) \\ & - 4\sigma_{12}^{(k+1)} s_{44}^{1(k+1)} \cos \theta_{k+1} \sin \theta_{k+1} - \sigma_{11}^{(k)} (s_{11}^{1(k)} \cos^2 \theta_k + s_{12}^{1(k)} \sin^2 \theta_k) \\ & + \sigma_{22}^{(k)} (s_{12}^{1(k)} \cos^2 \theta_k + s_{22}^{1(k)} \sin^2 \theta_k) = 4\sigma_{12}^{(k)} s_{44}^{1(k)} \cos \theta_k \sin \theta_k \\ & = - \sigma_{11}^{(k+1)} (s_{11}^{11(k+1)} \cos^2 \theta_{k+1} + s_{12}^{11(k+1)} \sin^2 \theta_{k+1}) \\ & + \sigma_{22}^{(k+1)} (s_{12}^{11(k+1)} \cos^2 \theta_{k+1} + s_{22}^{11(k+1)} \sin^2 \theta_{k+1}) - 4\sigma_{12}^{(k+1)} s_{44}^{11(k+1)} \cos \theta_{k+1} \sin \theta_{k+1} \\ & + \sigma_{11}^{(k)} (s_{11}^{11(k)} \cos^2 \theta_k + s_{12}^{11(k)} \sin^2 \theta_k) + \sigma_{22}^{(k)} (s_{12}^{11(k)} \cos^2 \theta_k + s_{22}^{11(k)} \sin^2 \theta_k) \\ & - 4\sigma_{12}^{(k)} s_{44}^{11(k)} \cos \theta_k \sin \theta_k \quad (3.1.15) \end{aligned}$$

$$\begin{aligned}
& \sigma_{11}^{(k+1)} (S_{11}^{1(k+1)} \sin^2 \theta_{k+1} + S_{12}^{1(k+1)} \cos^2 \theta_{k+1}) \\
& + \sigma_{22}^{(k+1)} (S_{12}^{1(k+1)} \sin^2 \theta_{k+1} + S_{22}^{1(k+1)} \cos^2 \theta_{k+1}) \\
& + 4\sigma_{12}^{(k+1)} S_{44}^{1(k+1)} \cos \theta_{k+1} \sin \theta_{k+1} \\
& \sigma_{11}^{(k)} (S_{11}^{1(k)} \sin^2 \theta_k + S_{12}^{1(k)} \cos^2 \theta_k) \\
& + \sigma_{22}^{(k)} (S_{12}^{1(k)} \sin^2 \theta_k + S_{22}^{1(k)} \cos^2 \theta_k) \\
& + 4\sigma_{12}^{(k)} S_{12}^{1(k)} S_{44}^{1(k)} \cos \theta_k \sin \theta_k \\
& = - \sigma_{11}^{(k+1)} (S_{11}^{11(k+1)} \sin^2 \theta_{k+1} + S_{12}^{11(k+1)} \cos^2 \theta_{k+1}) \\
& + \sigma_{22}^{(k+1)} (S_{12}^{11(k+1)} \sin^2 \theta_{k+1} + S_{22}^{11(k+1)} \cos^2 \theta_{k+1}) \\
& + 4\sigma_{12}^{(k+1)} S_{44}^{11(k+1)} \cos \theta_{k+1} \sin \theta_{k+1} + \sigma_{11}^{(k)} (S_{11}^{11(k)} \sin^2 \theta_k + S_{12}^{11(k)} \cos^2 \theta_k) \\
& + \sigma_{22}^{(k)} (S_{12}^{11(k)} \sin^2 \theta_k + S_{22}^{11(k)} \cos^2 \theta_k) + 4\sigma_{12}^{(k)} S_{44}^{11(k+1)} \cos \theta_{k+1} \sin \theta_{k+1}
\end{aligned} \tag{3.1.16}$$

$$\begin{aligned}
& \sigma_{11}^{(k+1)} (S_{11}^{1(k+1)} - S_{22}^{1(k+1)}) \sin \theta_{k+1} \cos \theta_{k+1} \\
& + \sigma_{22}^{(k+1)} (S_{12}^{1(k+1)} - S_{22}^{1(k+1)}) \sin \theta_{k+1} \cos \theta_{k+1} \\
& + 2\sigma_{12}^{(k+1)} S_{44}^{1(k+1)} (\cos^2 \theta_{k+1} - \sin^2 \theta_{k+1}) \\
& - \sigma_{11}^{(k)} (S_{11}^{1(k)} - S_{12}^{1(k)}) \sin \theta_k \cos \theta_k \\
& + \sigma_{22}^{(k)} (S_{12}^{1(k)} - S_{22}^{1(k)}) \sin \theta_k \cos \theta_k + 2\sigma_{12}^{(k)} S_{44}^{1(k)} (\cos^2 \theta_k - \sin^2 \theta_k) \\
& = - \sigma_{11}^{(k+1)} (S_{11}^{11(k+1)} - S_{12}^{11(k+1)}) \sin \theta_{k+1} \cos \theta_{k+1} \\
& + \sigma_{22}^{(k+1)} (S_{12}^{11(k+1)} - S_{22}^{11(k+1)}) \sin \theta_k \cos \theta_k \\
& + 2\sigma_{12}^{(k+1)} S_{44}^{11(k+1)} (\cos^2 \theta_{k+1} - \sin^2 \theta_{k+1}) \\
& + \sigma_{11}^{(k)} (S_{11}^{11(k)} - S_{12}^{11(k)}) \sin \theta_k \cos \theta_k \\
& + \sigma_{22}^{(k)} (S_{12}^{11(k)} - S_{22}^{11(k)}) \sin \theta_k \cos \theta_k \\
& + 2\sigma_{12}^{(k)} S_{44}^{11(k)} (\cos^2 \theta_k - \sin^2 \theta_k)
\end{aligned} \tag{3.1.17}$$

27.

$k = 1, 2, \dots, k-1$

To these must be adjoined equations (3.1.9) which are written here in components:

$$\sum_{k=1}^k (\sigma_{11}^{(k)} \cos^2 \theta_k + \sigma_{22}^{(k)} \sin^2 \theta_k - 2\sigma_{12}^{(k)} \cos \theta_k \sin \theta_k) t_k = \sigma_{11}^0 h \quad (a)$$

$$\sum_{k=1}^k (\sigma_{11}^{(k)} \sin^2 \theta_k + \sigma_{22}^{(k)} \cos^2 \theta_k + 2\sigma_{12}^{(k)} \cos \theta_k \sin \theta_k) t_k = \sigma_{22}^0 h \quad (b)$$

$$\sum_{k=1}^k (\sigma_{11}^{(k)} - \sigma_{22}^{(k)}) \cos \theta_k \sin \theta_k + \sigma_{12}^{(k)} (\cos^2 \theta_k - \sin^2 \theta_k) t_k = \sigma_{12}^0 h \quad (c)$$

(3.1.18)

We now consider special cases of interest. In the first case the inelastic laminae strains have the form (2.1.13). Then the right side of (3.1.15-.17) simplifies by setting:

$$\begin{aligned} s_{11}^{11(k)} &= s_{11}^{11(k+1)} = s_{12}^{11(k)} = s_{12}^{11(k+1)} = 0 \\ s_{22}^{11(k)} &= s_{22}^{11(k)} (\sigma_{22}^{(k)}, \sigma_{12}^{(k)}) \\ s_{22}^{11(k+1)} &= s_{22}^{11(k+1)} (\sigma_{22}^{(k+1)}, \sigma_{12}^{(k+1)}) \\ s_{44}^{11(k)} &= s_{44}^{11(k)} (\sigma_{22}^{(k)}, \sigma_{12}^{(k)}) \\ s_{44}^{11(k+1)} &= s_{44}^{11(k+1)} (\sigma_{22}^{(k+1)}, \sigma_{12}^{(k+1)}) \end{aligned} \quad (3.1.19)$$

Once the stresses in the laminae have been obtained the strains in the laminae, referred to laminate axes, are determined from (3.1.11). Since the strains in all laminae are the same when referred to the laminate coordinate system, these

are also the average laminate strains and thus determine the inelastic response of the laminate.

In the simplest case the lamina material is assumed to be inelastic in shear only. In that event we have in addition to (3.1.19):

$$s_{22}''^{(k)} = s_{22}''^{(k+1)} = 0 \quad (3.1.20)$$

and for Ramberg-Osgood presentation of inelastic part of shear compliance:

$$\begin{aligned} 2s_{44}^{11(k)} &= \frac{1}{2G_A^{(k)}} \left( \frac{\sigma_{12}^{(k)}}{\tau_Y^{(k)}} \right)^{N_k-1} \\ 2s_{44}^{11(k+1)} &= \frac{1}{2G_A^{(k+1)}} \left( \frac{\sigma_{12}^{(k+1)}}{\tau_Y^{(k+1)}} \right)^{N_{k+1}-1} \end{aligned} \quad (3.1.21)$$

In Ramberg-Osgood representation (2.2.1) the inelastic parts of the compliances assume forms such as:

$$\begin{aligned} s_{22}^{11(k)} &= \frac{1}{E_T^{(k)}} \left[ \left( \frac{\sigma_{22}^{(k)}}{\sigma_Y^{(k)}} \right)^2 + \left( \frac{\sigma_{12}^{(k)}}{\tau_Y^{(k)}} \right)^2 \right]^{1/2 (M_k-1)} \\ 2s_{44}^{11(k)} &= \frac{1}{2G_A^{(k)}} \left[ \left( \frac{\sigma_{22}^{(k)}}{\sigma_Y^{(k)}} \right)^2 + \left( \frac{\sigma_{12}^{(k)}}{\tau_Y^{(k)}} \right)^2 \right]^{1/2 (N_k-1)} \\ s_{22}^{11(k+1)} &= \frac{1}{E_T^{(k+1)}} \left[ \left( \frac{\sigma_{22}^{(k+1)}}{\sigma_Y^{(k+1)}} \right)^2 + \left( \frac{\sigma_{12}^{(k+1)}}{\tau_Y^{(k+1)}} \right)^2 \right]^{1/2 (M_{k+1}-1)} \\ 2s_{44}^{11(k+1)} &= \frac{1}{2G_A^{(k+1)}} \left[ \left( \frac{\sigma_{22}^{(k+1)}}{\sigma_Y^{(k+1)}} \right)^2 + \left( \frac{\sigma_{12}^{(k+1)}}{\tau_Y^{(k+1)}} \right)^2 \right]^{1/2 (N_{k+1}-1)} \end{aligned} \quad (3.1.22)$$

### 3.2 Method of Solution

The equations which define the laminae stresses are (3.1.9) and (3.1.13) in condensed form, or equivalently, (3.1.15- 3.1.17), (3.1.18) in full form. To explain the solution method it is simpler to write in terms of the condensed form.

Define the matrices:

$$\begin{aligned}\underline{L}^{1(k+1)} &= \underline{\theta}^{(k+1)} \underline{S}^{1(k+1)} \\ \underline{L}^{11(k+1)} &= \underline{\theta}^{(k+1)} \underline{S}^{11(k+1)} \\ \underline{L}^{1(k)} &= \underline{\theta}^{(k)} \underline{S}^{1(k)} \\ \underline{L}^{11(k)} &= \underline{\theta}^{(k)} \underline{S}^{11(k)}\end{aligned}\tag{3.2.1}$$

Then equs. (3.1.13) assume the form:

$$\underline{L}^{1(k+1)} \underline{\sigma}^{(k+1)} - \underline{L}^{1(k)} \underline{\sigma}^{(k)} = -\underline{L}^{11(k+1)} \underline{\sigma}^{(k+1)} + \underline{L}^{11(k)} \underline{\sigma}^{(k)}\tag{3.2.2}$$

to which are adjoined equs. (3.1.9) which are here rewritten:

$$\sum_{k=1}^K \underline{\theta}^{(k)} \underline{\sigma}^{(k)} = \underline{\sigma}^o\tag{3.2.3}$$

The equations may be solved numerically by an iteration method which proceeds as follows: Consider equs. (3.2.2-3) with the right side of (3.2.2) zero. This defines a set of stresses  $\underline{\sigma}_o^{(k)}$  given by:

$$\begin{aligned}\underline{L}^{1(k+1)} \underline{\sigma}_o^{(k+1)} - \underline{L}^{1(k)} \underline{\sigma}_o^{(k)} &= 0 \\ k &= 1, 2, \dots, K\end{aligned}\tag{3.2.4}$$

$$\sum_{k=1}^K \underline{\theta}^{(k)} \underline{\sigma}_o^{(k)} = \underline{\sigma}^o\tag{b}$$

Since (3.2.4a) contains only elastic compliances  $S^{(k)}$  it is seen that the equations are linear and define the stresses in an elastic laminate. Now insert the stresses  $\underline{\sigma}_o^{(k)}$  into the right side of (3.2.2) and define the stresses  $\underline{\sigma}_1^{(k)}$  by:

$$\begin{aligned} \underline{L}^{1(k+1)} \underline{\sigma}_1^{(k+1)} - \underline{L}^{1(k)} \underline{\sigma}_1^{(k)} = & - \underline{L}^{11(k+1)} [\underline{\sigma}_o^{(k+1)}] \underline{\sigma}_o^{(k+1)} \quad (a) \\ & + \underline{L}^{11(k)} [\underline{\sigma}_o^{(k)}] \underline{\sigma}_o^{(k)} \end{aligned} \quad (3.2.5)$$

$$\sum_{k=1}^K \underline{\theta}^{(k)} \underline{\sigma}_1^{(k)} = \underline{\sigma}^o \quad (b)$$

Eqs. (3.2.5) defines (hopefully) a new approximation  $\underline{\sigma}_1^{(k)}$  which is the solution of a set of linear equations. The stresses in square brackets in the right side of (3.2.5) are to emphasize the stress dependence of the non-linear parts of the compliances.

The procedure just initiated can be repeated indefinitely. In general:

$$\begin{aligned} \underline{L}^{1(k+1)} \underline{\sigma}_{\ell+1}^{(k+1)} - \underline{L}^{1(k)} \underline{\sigma}_{\ell+1}^{(k)} = & - \underline{L}^{11(k+1)} [\underline{\sigma}_\ell^{(k+1)}] \underline{\sigma}_\ell^{(k+1)} \\ & + \underline{L}^{11(k)} [\underline{\sigma}_\ell^{(k)}] \underline{\sigma}_\ell^{(k)} \end{aligned} \quad (3.2.6)$$

$$\sum_{k=1}^K \underline{\theta}^{(k)} \underline{\sigma}_{\ell+1}^{(k)} = \underline{\sigma}^o$$

This iteration procedure is quite easy to carry out with aid of a computer. It replaces the solution of a set of non-linear equations by solution of a sequence of linear equations, provided of course, that convergence is obtained.

It should be noted that the first iteration step does not necessarily have to start with equs. (3.2.4a), i.e., with zero right side of (3.2.2). Any stresses  $\underline{\sigma}_o^{(k)}$  which fulfill (3.2.4b) can be used to start the iteration with (3.2.5) and continuing with the general iteration relation (3.2.6).

It is desired to obtain a laminate solution for only one load system  $\underline{\sigma}^o$  then it would seem most logical to start with (3.2.4). But suppose there is a sequence of loadings  $\Delta \underline{\sigma}^o$ ,  $2\Delta \underline{\sigma}^o \dots n\Delta \underline{\sigma}^o$ . Suppose that a solution for  $(n-1) \Delta \underline{\sigma}^o$  has been obtained and that a solution for  $n\Delta \underline{\sigma}^o$  is desired. One possibility is to multiply all stresses due to the load  $(n-1) \Delta \underline{\sigma}^o$  by the factor  $n/(n-1)$ . The stresses thus obtained certainly

also satisfy (3.2.6b) because of the linearity of these equations. They will generally be reasonable starting values  $\sigma_o^{(k)}$  for the iteration.

This method of iteration to obtain a solution was found to work well for many sample problems; however, there were cases in which the solution did not converge. Attempts to modify the recurrence relations to overcome this problem met with only partial success. Thus, an alternate procedure for solution was defined. The solution was obtained by application of the Newton-Raphson method.

The set of  $3K$  nonlinear equations represented by equs. (3.2.2-.3) may be presented in the form:

$$F_n(\sigma_{ij}^k) = 0 \quad n = 1, 2 \dots 3K \quad (3.2.7)$$

The function  $F_i$  is expanded in a Taylor series about an arbitrary set of initial stresses which may be taken as the solutions of the elasticity problem. Considering only two terms of the series, it is found that

$$F_i = F_i^o + \frac{\partial F_i^o}{\partial \sigma_{mn}^k} \Delta \sigma_{mn}^k = 0 \quad (3.2.8)$$

or

$$\sigma_{ij}^k = {}^o\sigma_{ij}^k - \left[ \frac{\partial F_m^o}{\partial \sigma_{ij}^k} \right]^{-1} F_m^o \quad (3.2.9)$$

where  $\sigma_{ij}^k$  is the corrected solution obtained from the assumed solution  ${}^o\sigma_{ij}^k$ . Using  $\sigma_{ij}^k$  as the initial guess, the process is repeated until the result is obtained within a desired accuracy. A recurrence form of equation (3.2.9) to obtain the stresses at  $t+1$  cycle from  $t$  cycle can be constructed as follows:

$$(\sigma_{ij}^k)_{t+1} = (\sigma_{ij}^k)_t - \left[ \frac{\partial F_m}{\partial \sigma_{ij}^k} \right]^{-1} (F_m)_t \quad (3.2.10)$$

After the stresses  $\sigma_{ij}^k$  are obtained for all layers of the laminate, strains for any layer  $k$  in terms of laminae axes can be computed using eqs. (3.1.3). Strains in terms of the laminate axes can be obtained using the strain transformation law.

This analysis has been developed into an efficient computer program. A description of the program including a listing, is presented in Appendix E.



### 3.3 Numerical Results

The computer program which has been developed under the present study has been utilized in the analysis of a variety of different composite laminates. The initial studies using the computerized analysis were directed at presenting a comparison between the results of the present analysis and those of previous analyses, notably that of Ref. 9. (The present results were also compared to available experimental data, primarily those of Ref. 6 which had also been used for comparison with the analytical results in Ref. 9.) The objective of this phase of the numerical study was to determine whether the present results, which can be obtained with minimal computer usage, compare well with those of the more exact and complex analytical results in Ref. 9. The results of this comparison are highly encouraging, as will be shown below, and support the utilization of the present analysis as an efficient design tool.

In the second phase of the design numerical studies, consideration was given to examining the sensitivity of laminate results to individual properties of the layers. These parametric studies are presented for several classes of typical laminates.

A series of laminates of boron/epoxy composites for which experimental data had been obtained in Ref. 6 were examined analytically in Ref. 9. In Figures 9 to 15, results of the present analytical method are added to the comparison of experimental results of [6] and analytical results of [9]. For example, in Fig. 9, the experimental stress-strain curve for a 0-90 boron/epoxy laminate is compared to the analytical results obtained in Ref. 9 and in the present analysis. Both analytical results coincide; both show slightly less inelastic strain than the experiment. The solid point on the curve indicates the stress level at which fiber fracture is computed to occur in one of the layers of the laminate.

The shear stress-strain curve used in the present analysis was the best fit Ramberg-Osgood curve having an exponent  $n=3$ .

The values of modulus and yield stress obtained from the least squares fit are shown on the figure. A similar result is shown for the unidirectional tension  $\pm 45^\circ$  laminate in Fig. 10. Here it is seen that the two analytical curves are similar, although the agreement is not as close as in Fig. 9. Experimental data reflect a substantially higher degree of inelasticity than either analytical result. The present analysis shows a higher degree of inelastic strain at the higher stress level than that of Ref. 9. However, the reverse is true in the comparison of the two analytical results shown in Fig. 11 for a  $\pm 30^\circ$  laminate. The present results were obtained with a linear stress-strain curve in the transverse direction within each of the layers. The computations were made in this fashion because the transverse stress-strain curve of Ref. 9 does not show a significant degree of inelasticity.

Figure 12 presents results for the case of a quasi-isotropic laminate (0/ $\pm 45/90$ ) of boron/epoxy. Both the present result and that of Ref. 9 show a relatively insignificant amount of inelasticity. Again, the experimental data show a greater inelastic effect. Here the predicted failure strain level is in good agreement with the experimental failure strain level; however, there is a significant difference in the failure stress level. A similar result is presented in Fig. 13 for the quasi-isotropic laminate formed from the 0/ $\pm 60^\circ$  configuration.

Computations performed for the present study for laminates having fibers in several directions, including the loading direction, for a simple unidirectional load have shown a relatively small amount of inelastic strain. Another example of this is presented in Fig. 14 for a 0/ $\pm 45^\circ$  laminate. Here, however, the agreement of all the analytical methods and the experimental method is very good.

The final comparison taken from Ref. 9 is presented in Fig. 15 for a laminate having fibers in three different directions and a tensile load applied at some intermediate angle. The present analysis agrees reasonably well with the results

Ref. 9. The discrepancy between the failure load predicted on the basis of fiber failure and the experimentally observed failure stress is quite substantial. It is possible that failure in laminate of this type could result from shearing or transverse stresses within the individual layers, and thus, not be a result of tension in the fiber failure. This mode of failure has not been treated in the present computer program. The mode of failure observed experimentally is not known to the authors.

The experimentally measured response of a multidirectional laminate to an applied shear stress has been reported in Ref. 13. Comparison of the experimental result with the theory of Ref. 9 was presented in Ref. 14. Computations for this case, made using the present analysis and the prior analytical result (Ref. 14), are compared to the experimental result in Fig. 16. Again, correlation between the two analytical results is good, agreement between analytical and theoretical results is reasonably good with the experimental observation showing higher inelastic strains and lower tangent shear moduli at the very high stress levels.

The conclusion of these comparisons with analytical and experimental data seem to justify the adoption of the present computer program as a useful engineering tool for the design and analysis of composite laminates. However, it appears that further study of the failure region is required.

Parametric study of the influence of various laminate geometric and mechanical properties has also been explored. Fig. 17 shows the results obtained for a  $0/\pm 45^\circ$  laminate indicating that the inelastic response in the transverse direction can become significant at higher stress levels. Failure due to fiber fracture under a transverse stress applied to the laminate occurs at strain levels larger than those plotted in Fig. 17. In the quasi-isotropic laminate having four fiber directions,  $(0/\pm 45/90)$  the degree of inelasticity in the longitudinal and transverse directions is of course the same and is

in both cases very small. It is to be anticipated, on an intuitive basis, that the maximum degree of inelastic response would be observed for a stress applied midway between two of the fiber directions on this quasi-isotropic laminate. The stress-strain curve for this latter case is also shown in Fig. 18. Although the inelastic strains for this case are not significant there is a large difference in the predicted failure stress levels based on stress in the fiber direction for the two cases. It is worthwhile to emphasize that the quasi-isotropic laminate need not be isotropic in its strength characteristics.

Because of the directional strength characteristics interesting effects may be expected for combined stress cases. Some results of the exploration of this question are presented in Fig. 19 where the four direction quasi-isotropic laminate is subjected to combined stress state with respect to a  $22\text{-}1/2^\circ$  axis of symmetry. This laminate shows high strength under both the unidirectional load and shear load by itself. The combined stress case for equal values of applied shear stress and axial stress results in fiber failure, and therefore, laminate failure, at a substantially lower stress. The stress-strain curve prior to failure is not affected significantly by the presence of combined stress. The quasi-isotropic laminate having fibers in three directions ( $0/+60$ ) is examined in Fig. 20. The sensitivity of this laminate to the Ramberg-Osgood parameters for the individual ply had little effect upon the stress-strain result. Indeed as an extreme example of this variation all laminates stiffnesses except the axial stiffness were equal to zero. Enforcement of the Kirchhoff-Love plate assumptions for this case results in the so-called netting analysis. The response for this netting case, which is linear, is shown by the dashed curve in Fig. 20. Even with this extreme assumption, matrix inelasticity does not introduce a significant amount of inelastic strain. Experimental data for comparison with this result are not easily available, however Ref. 17 does present a stress-strain curve for this case which shows a transverse failure stress for the

quasi-isotropic  $0/\pm 60^\circ$  laminate which is about 60% of the failure stress in the axial direction. Also, the inelastic strain at failure is approximately 30% larger than the elastic strain associated with the failure stress level. The netting analysis result presented here suggests that in order to obtain such a strain, one might have to consider that the axial stiffness, either in tension, compression or both; or that other effects not considered in the conventional laminate analysis, such as interlaminar or transverse shear deformations, might contribute significantly to the overall laminate deformation.

The influence of the characteristic stress levels for transverse stress and axial shear of the unidirectional layer of a boron/epoxy material is examined in Fig. 21. The measure of this effect is taken to be the influence upon the stress-strain curve for the unidirectional tension of  $\pm 30^\circ$  laminate. The strong sensitivity to the characteristic axial shear stress  $\tau_y$  and the relative insensitivity to the transverse characteristic stress  $\sigma_y$  for the R-O representations is illustrated in the figure. A similar comparison made for a boron/aluminum laminate of the same geometry subjected to uniaxial applied stress is shown in Fig. 22. Similar sensitivities are observed for this case. Boron/aluminum laminate response under transverse applied stress with the same values of the Ramberg-Osgood parameters is shown in Fig. 23. Here the fiber failure criterion did not come into play and thus the computations were extended to rather large strains in matrix. It is clear, that for this case, the failure criterion based on other stress-strain components is required. The examination of the computer print-out permits one to terminate the stress-strain curves at some stress level prior to fiber fracture depending upon the choice of the failure criterion. This can be done rather readily. The choice of the failure criterion is discussed in Appendix D.

The lamina properties for boron/aluminum are used to

analyze a  $0^\circ/\pm 30^\circ$  laminate under combined loading. These results are shown in Fig. 24. Axial stress-strain curves are presented for varying ratios of axial shear stress to axial tensile stress.

#### 4. CONCLUDING REMARKS

Current approaches to the definition of design allowable stress for advanced fiber composite laminates are based upon the utilization of extremely conservative criteria. These limit the laminate to stress levels below which no significant damage of any kind occurs. The utilization of overly conservative design criteria can negate much of the potential for effective design utilizing advanced composite materials. The heterogeneous nature of these materials is such that a variety of possible damage modes exist. Thus, matrix cracking or yielding, fiber fracture, debonding, and other inelastic effects can all occur in local regions at relatively low average stress levels. These nonuniform and nonlinear effects greatly complicate the problem of establishing reliable design allowables. In the present program, the problem of nonlinear laminate behavior resulting from nonlinearities in the behavior of the matrix material was studied. The objective of the program was to develop an understanding of the inelastic behavior of composite laminates and to develop a computer program which will be used as an engineering tool in the design of fiber composite laminated structures.

The method of approach utilized herein was to adopt a Ramberg-Osgood representation of the nonlinear stress-strain behavior and to utilize deformation theory as an adequate representation of the material nonlinearities. The problem was viewed on two levels. First, the relationship between the constituent properties and the stress-strain response of a unidirectional fiber composite material was studied. For this problem, the primary attention herein was directed toward the axial shear behavior, in as much as experimental data had indicated that it is this type of load which results in the most significant nonlinearities in material behavior. For this case, an expression was established relating the composite average-stress/average-strain curve to the fiber moduli and the matrix nonlinear stress-strain curve. This expression, which was developed as a lower bound, was found to give good agreement with the more exact results obtained by

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applying incremental plasticity theory and using a numerical finite element analysis to the assessment of the material behavior (Ref. 9).

The second level of approach treats the interrelationship between the properties of the unidirectional layers and those of the laminate. For this case, one may consider that the starting point is a nonlinear stress-strain curve for transverse stress, and for axial shear stress, alone, and a linear stress-strain relation for stress in the fiber direction. The nonlinear lamina stress-strain curves can be modeled by proper selection of the Ramberg-Osgood parameters.

In the present study, unlike other formulations an interaction expression was formulated to account for simultaneous application of axial shear and transverse stress. A laminate having an arbitrary number of oriented layers, and subjected to a general state of membrane stress, was treated. The results of this analysis were programmed into an efficient computer routine for numerical evaluation of arbitrary laminates. Results obtained show good agreement with those of previous complex numerical methods utilizing incremental plasticity theory.

Certain limitations connected with this program should also be discussed. First, deformation type stress-strain relations have been used; hence, it is implicit in this result that the stress and strain values obtained for any given set of loads are functions only of those loads and not of the loading history. On the other hand, if points are computed for intermediate values of loads, following different load paths, then different intermediate conditions will be obtained. Thus, the question is raised as to what is the accuracy of the results obtained for paths which do not yield proportional loading. It is known that for local proportional loading, the deformation theory result is the same as that for the incremental theory. In the laminate, local proportional loading does not exist, in general, even when the external loading is proportional. However, the assumption is made that the deformation theory will yield an approximation which is satisfactory to generate a



rational engineering tool. This can only be assessed by comparison with an exact analysis, or since this does not exist for the case of arbitrary loading paths, perhaps by comparison with experimental data.

Comparisons of the present results with experimental data tend to show moderately good agreement. There are, however, cases in which experimental results show a higher degree of inelastic strain than predicted by the present analysis. These experimental data are quite limited and may be insufficient for drawing conclusions in this regard.

The question of failure criteria incorporated into the present analysis required further consideration. The present analysis obtains more accurate representations of the stress components in the individual layers than have been obtained from elastic analyses. Hence, the use of these stress components in any failure criteria should represent an improvement in failure prediction.

In addition to a description of the methods of analysis, and of the numerical comparisons which have been carried out, the present report also presents a description of the computer program for study of nonlinear behavior of laminates in sufficient detail to permit the utilization of this program by others.

## APPENDIX A

### SYMMETRY SIMPLIFICATION OF NON-LINEAR STRESS-STRAIN RELATIONS

The most general inelastic stress-strain relations of the deformation type are of the form

$$\epsilon_{ij} = S_{ijkl} \sigma_{kl} \quad (1)$$

where  $S_{ijkl}$  are functions of the stresses. Let it be assumed that the material is transversely isotropic with  $x_1$  axis of symmetry. Any rotation about  $x_1$  changes  $\epsilon_{ij}$  and  $\sigma_{ij}$  into  $\epsilon'_{ij}$  and  $\sigma'_{ij}$ . Then the condition of transverse isotropy demands that

$$\epsilon'_{ij} = S_{ijkl} \sigma'_{kl} \quad (2)$$

where  $S_{ijkl}$  in (1) and (2) are the same. To fulfill this last requirement it is necessary that  $S_{ijkl}$  be functions of stresses only through stress expressions which are invariant for rotations about the  $x_1$  axis. There are five such invariants and they are given by, [15]

$$\begin{aligned} I_1 &= \sigma_{11} & I_2 &= \sigma_{22} + \sigma_{33} & I_3 &= \sigma_{12}^2 + \sigma_{13}^2 & (3) \\ I_4 &= 1/2(\sigma_{22} - \sigma_{33})^2 + 2\sigma_{23}^2 & I_5 &= 1/2(\sigma_{22} - \sigma_{33})(\sigma_{12}^2 - \sigma_{13}^2) + 2\sigma_{12}\sigma_{13}\sigma_{23} \end{aligned}$$

Thus

$$S_{ijkl} = S_{ijkl} (I_1, I_2, I_3, I_4, I_5) \quad (4)$$

It follows that for rotations around the  $x_1$  axis of symmetry the  $S_{ijkl}$  behave as constants. Consequently, the symmetry reduction of (1) to transverse isotropy is just as in elasticity.

The reduction may be performed in following fashion: For rotation of angle  $\theta$  about the  $x_1$  axis, the stress tensor  $\sigma_{ij}$  transforms into  $\sigma'_{ij}$  in the following fashion

$$\begin{aligned} \sigma'_{11} &= \sigma_{11} \\ \sigma'_{22} &= 1/2 (\sigma_{22} + \sigma_{33}) + 1/2 (\sigma_{22} - \sigma_{33}) \cos 2\theta + \sigma_{23} \sin 2\theta \\ \sigma'_{33} &= 1/2 (\sigma_{22} + \sigma_{33}) - 1/2 (\sigma_{22} - \sigma_{33}) \cos 2\theta - \sigma_{23} \sin 2\theta \\ \sigma'_{23} &= 1/2 (\sigma_{33} - \sigma_{22}) \sin 2\theta + \sigma_{23} \cos 2\theta \\ \sigma'_{12} &= \sigma_{12} \cos \theta + \sigma_{13} \sin \theta \\ \sigma'_{13} &= -\sigma_{12} \sin \theta + \sigma_{13} \cos \theta \end{aligned} \quad (5)$$

The same transformation relations obviously also hold for strains. If the transformed stresses and strains are introduced into (2) then coefficients of  $\cos 2\theta$ ,  $\sin 2\theta$ ,  $\cos \theta$  and  $\sin \theta$  and remaining terms independent of  $\theta$  must be equal. These equalities result in relations among the various components which reduce the stress-strain law to the form (2.1.4- 5) from Chapter 2 of this report. (Average stresses and strains appear in the latter but this obviously makes no differences in the derivation.)

## APPENDIX B

### PLANE STRESS-STRAIN RELATIONS OF FIBER REINFORCED MATERIAL IN GENERALIZED RAMBERG-OSGOOD FORM

The purpose of the present appendix is to arrive at equs. (2.2.7). For convenience in writing, overbars on stresses and strains will be omitted.

The present development is guided by isotropic  $J_2$  theory for deformation type plastic stress-strain relations. The basic assumption of this theory in the isotropic case is that the plastic strains have the form

$$\epsilon_{ij}'' = f(J_2) s_{ij} \quad (1)$$

where  $s_{ij}$  is the stress deviator and

$$J_2 = 1/2 s_{ij} s_{ij} \quad (2)$$

is its second invariant.

It is instructive to work out the form of (1) for Ramberg-Osgood type stress-strain relations. Suppose that in pure shear the stress-strain relation is

$$\epsilon_{12}'' = \frac{\sigma_{12}}{2G} \left[ 1 + \left( \frac{\sigma_{12}}{\tau_Y} \right)^{n-1} \right] \quad (3)$$

Now in pure shear it follows from (2) that

$$J_2 = \sigma_{12}^2$$

Therefore (3) can be written in the form

$$\epsilon_{12}'' = \frac{\sigma_{12}}{2G} \left[ 1 + \left( \frac{\sqrt{J_2}}{\tau_Y} \right)^{n-1} \right] \quad (4)$$

which is in the form (1). Consequently, in the general case of three dimensional stress and strain

$$\epsilon_{ij}'' = \frac{s_{ij}}{2G} \left[ 1 + \left( \frac{\sqrt{J_2}}{\tau_Y} \right)^{n-1} \right] \quad (5)$$

It should be emphasized that there is nothing fundamental about (1). It is an assumption which states that the plastic strains can be represented by the stress deviator components multiplied by a function of a quadratic expression in the stresses which is  $J_2$ . The choice of  $J_2$  for a quadratic expression is not arbitrary but may be arrived at by isotropy arguments.

In an anisotropic material it may be assumed by generalization that plastic strains are given by

$$\epsilon_{ij}'' = s_{ij}'' f(L) \quad (6)$$

Where  $L$  is some general quadratic function of the stresses. This assumption will form the basis of the present development.

Consider the stress-strain relations (2.1.13). It is assumed that  $s_{22}''$  and  $s_{44}''$  functions of the most general quadratic form in  $\bar{\sigma}_{22}$  and  $\bar{\sigma}_{12}$ .

Thus

$$\begin{aligned} s_{22}'' &= s_{22}'' (A\bar{\sigma}_{22}^2 + B\bar{\sigma}_{22}\bar{\sigma}_{12} + C\bar{\sigma}_{12}^2) \\ s_{44}'' &= s_{44}'' (A\bar{\sigma}_{22}^2 + B\bar{\sigma}_{22}\bar{\sigma}_{12} + C\bar{\sigma}_{12}^2) \end{aligned} \quad (7)$$

It should be noted that the material reacts in same fashion to positive or negative shear stress, therefore also in same fashion to some  $\bar{\sigma}_{22}$  together with positive or negative shear stress. However, the middle term in the quadratic changes sign with shear stress. Therefore, this term should be omitted.

Now rewrite (7) in form

$$\begin{aligned} s_{22}'' &= \frac{1}{E_T} f_{22} (\alpha^2 \bar{\sigma}_{22}^2 + \beta^2 \bar{\sigma}_{12}^2) \\ s_{44}'' &= \frac{1}{2G_T} f_{44} (\alpha^2 \bar{\sigma}_{22}^2 + \beta^2 \bar{\sigma}_{12}^2) \end{aligned} \quad (8)$$

where  $f_{22}$  and  $f_{44}$  are nondimensional functions and  $\alpha$  and  $\beta$  have dimensions of reciprocal of stress. If  $\bar{\sigma}_{12}=0$  the first of (8) assumes the form

$$s_{22}'' = \frac{1}{E_T} f_{22} (\alpha^2 \bar{\sigma}_{22}^2) \quad (9)$$

For one dimensional  $\bar{\sigma}_{22}$ , from the Ramberg-Osgood stress-strain relation (2.2.6a)

$$s_{22}'' = \frac{1}{E_T} \left( \frac{\bar{\sigma}_{22}}{\sigma_Y} \right)^{M-1}$$

which can be written as

$$s_{22}'' = \frac{1}{E_T} \left[ \left( \frac{\bar{\sigma}_{22}}{\sigma_Y} \right)^2 \right]^{\frac{M-1}{2}} \quad (10)$$

It follows from (8) and (10) that

$$\alpha^2 = \frac{1}{\sigma_Y^2} \quad (11)$$

and the function of  $f_{22}$  is determined as  $(M-1)/2$  power.

In similar fashion, when  $\bar{\sigma}_{22}=0$ , the second of (8) assumes the form

$$s_{44}'' = \frac{1}{2G_T} f_{44} (\beta^2 \bar{\sigma}_{12}^2) \quad (12)$$

From the Ramberg-Osgood relation (2.2.6b) for one dimensional  $\bar{\sigma}_{12}$

$$s_{44}'' = \frac{1}{2G_T} \left( \frac{\bar{\sigma}_{12}}{\tau_Y} \right)^{N-1}$$

which can be written as

$$s_{44}'' = \frac{1}{2G_T} \left[ \left( \frac{\bar{\sigma}_{12}}{\tau_Y} \right)^2 \right]^{\frac{N-1}{2}} \quad (13)$$

It follows from (12) and (13) that

$$\beta^2 = \frac{1}{\tau_Y^2} \quad (14)$$

and the function  $f_{44}$  is determined as  $(N-1)/2$  power.

Consequently (8) now assumes the form

$$\begin{aligned} s''_{22} &= \frac{1}{E_T} \left[ \left( \frac{\bar{\sigma}_{22}}{\sigma_Y} \right)^2 + \left( \frac{\bar{\sigma}_{12}}{\tau_Y} \right)^2 \right] \frac{M-1}{2} \\ s''_{44} &= \frac{1}{2G_T} \left[ \left( \frac{\bar{\sigma}_{22}}{\sigma_Y} \right)^2 + \left( \frac{\bar{\sigma}_{12}}{\tau_Y} \right)^2 \right] \frac{N-1}{2} \end{aligned} \quad (15)$$

Then (2.2.7) follows from (15) and (2.1.13).

## APPENDIX C

### 1. EXTREMUM PRINCIPLES OF DEFORMATION THEORY OF PLASTICITY

#### i. Principle of Minimum Potential Energy

Let

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} \quad (1.1)$$

where  $C_{ijkl}$  are functions of the strains. The strain energy density is defined by the path dependent integral

$$W^\epsilon = \int_{\underline{\epsilon}=0}^{\underline{\epsilon}} \sigma_{ij}(\underline{\epsilon}) d\epsilon_{ij} \quad (1.2)$$

where  $\underline{\epsilon}$  is a concise notation for  $\epsilon_{ij}$ . The strain energy  $U^\epsilon$  of a body of volume  $V$  is defined by

$$U^\epsilon = \int_V W^\epsilon dV \quad (1.3)$$

Let the surface of the body be subjected to the boundary conditions

$$u_i(S) = u_i^0 \text{ on } S_u \quad (1.4)$$

$$T_i(S) = T_i^0 \text{ on } S_T$$

and let the body forces vanish. The potential energy  $U_p$  is defined by

$$U_p = \int_V W^\epsilon dV - \int_{S_T} T_i^0 u_i \quad (1.5)$$

Define an admissible displacement field  $\tilde{u}_i(\underline{x})$  by

$$\tilde{u}_i = u_i^0 \text{ on } S_u$$

$$\tilde{u}_i(\underline{x}) \text{ continuous everywhere} \quad (1.6)$$

Associated with  $\tilde{u}_i$  are the strains  $\tilde{\epsilon}_{ij}$  derived from it by the usual relations.



Define  $\tilde{W}^\epsilon$  by

$$\tilde{W}^\epsilon = \int_{\tilde{\epsilon}=0}^{\tilde{\epsilon}} \tilde{\sigma}_{ij} d\tilde{\epsilon}_{ij} \quad (1.7)$$

where

$$\tilde{\sigma}_{ij} = C_{ijkl}(\tilde{\epsilon}) \tilde{\epsilon}_{kl} \quad (1.8)$$

Define

$$\tilde{U}_P = \int_V \tilde{W}^\epsilon dV - \int_{S_T} T^\circ_i \tilde{u}_i dS \quad (1.9)$$

The principle of minimum potential energy for the present case then states that

$$\tilde{U}_P \geq U_P \quad (1.10)$$

equality taking place if and only if

$$\tilde{u}_i = u_i$$

In the event that displacements are prescribed over the entire surface, the surface integral in (1.9) vanishes. Then the principle reduces to that of minimum strain energy

$$\tilde{U}^\epsilon \geq U^\epsilon \quad (1.11)$$

## ii. Principle of Minimum Complementary Energy

Let

$$\epsilon_{ij} = S_{ijkl}(\underline{\sigma}) \sigma_{kl} \quad (1.12)$$

where  $S_{ijkl}$  are stress dependent compliances

Define the complementary energy density  $w^\sigma$  by the path dependent integral

$$w^\sigma = \int_{\underline{\sigma}=0}^{\underline{\sigma}} \varepsilon_{ij} d\sigma_{ij} \quad (1.13)$$

Let the surface of the body be subjected to the boundary conditions (1.4) and let the body forces vanish. The complementary energy  $U_C$  is defined by

$$U_C = \int_V w^\sigma dV - \int_{S_u} T_i u_i^\circ dS \quad (1.14)$$

Define an admissible stress field  $\tilde{\sigma}_{ij}$  by the following requirements

$$\tilde{\sigma}_{ij,j} = 0$$

$$\tilde{T}_i = \tilde{\sigma}_{ij} n_j \quad \text{continuous everywhere} \quad (1.15)$$

$$T_i(S) = T_i^\circ \text{ on } S_T$$

Define the complementary energy functional  $U_C$  by

$$\tilde{U}_C = \int_V \tilde{w}^\sigma dV - \int_{S_u} \tilde{T}_i u_i^\circ dS \quad (1.16)$$

$$\text{where } \tilde{w}^\sigma = \int_{\underline{\tilde{\sigma}}=0}^{\underline{\tilde{\sigma}}} \tilde{\varepsilon}_{ij} d\tilde{\sigma}_{ij} \quad (1.17)$$

$$\tilde{\varepsilon}_{ij} = S_{ijkl}(\underline{\tilde{\sigma}}) \tilde{\sigma}_{kl}$$

Then the principle of minimum complementary energy states that

$$U_C \geq \tilde{U}_C \quad (1.18)$$

equality occurring if and only if

$$\tilde{\sigma}_{ij} = \sigma_{ij}$$

If tractions are prescribed over the entire surface,  $S_u=0$ , the principle reduces to

$$\tilde{U}^\sigma \geq U^\sigma \quad (1.19)$$

For proof of these principles see e.g. [16]. An interesting application to obtain approximate solutions has been given in [17].

iii. Specialization of the Principles to Axial Shear with Ramberg-Osgood Stress-Strain Relations

In the case of axial shear of a uniaxially fiber reinforced material the only surviving stresses are

$$\sigma_{12} = \tau_2 \quad \sigma_{13} = \tau_3 \quad (1.20)$$

where 1 indicates fiber direction. Denote the associated shear strains by

$$\epsilon_{12} = \epsilon_2 \quad \epsilon_{13} = \epsilon_3 \quad (1.21)$$

Then the generalized Ramberg-Osgood stress-strain relations, Appendix B, (5) assume in the present case the form

$$\begin{aligned} \epsilon_2 &= \frac{\tau_2}{2G} \left[ 1 + \left( \frac{\tau}{\tau_y} \right)^{n-1} \right] \\ \epsilon_3 &= \frac{\tau_3}{2G} \left[ 1 + \left( \frac{\tau}{\tau_y} \right)^{n-1} \right] \\ \tau &= \sqrt{\tau_2^2 + \tau_3^2} \quad \sqrt{J_2} \end{aligned} \quad (1.22)$$

In the present case

$$\sigma_{ij} d\epsilon_{ij} = 2(\tau_2 d\epsilon_2 + \tau_3 d\epsilon_3) \quad (1.23)$$

Inserting (1.22) into (1.23) and using the relation

$$\tau d\tau = \tau_2 d\tau_2 + \tau_3 d\tau_3$$

it is easily shown that

$$\sigma_{ij} d\epsilon_{ij} = \frac{\tau}{G} \left[ 1 + n \left( \frac{\tau}{\tau_Y} \right)^{n-1} \right] d\tau \quad (1.24)$$

To compute  $W^\epsilon$  as defined by (1.2) it is necessary to integrate (1.24) from zero to some state of strain  $\epsilon_2, \epsilon_3$ . But it should be noted that (1.24) is expressed in terms of the variable  $\tau$  only. Now  $\tau$  can be expressed in terms of strains in following fashion. Define

$$\hat{\epsilon} = \sqrt{\epsilon_2^2 + \epsilon_3^2} \quad (1.25)$$

It follows at once from (1.22) that

$$\hat{\epsilon} = \frac{\tau}{2G} \left[ 1 + \left( \frac{\tau}{\tau_Y} \right)^{n-1} \right] \quad (1.26)$$

This relation defines  $\tau$  as a function of  $\hat{\epsilon}$ . Consequently,  $W^\epsilon$  assumes the form

$$W^\epsilon = \frac{1}{G} \int_0^{\tau(\hat{\epsilon})} \tau \left[ 1 + n \left( \frac{\tau}{\tau_Y} \right)^{n-1} \right] d\tau$$

which is easily integrated to yield

$$W^\epsilon = \frac{\tau^2}{2G} \left[ 1 + \frac{2n}{n+1} \left( \frac{\tau}{\tau_Y} \right)^{n-1} \right] \quad (1.27)$$

$$\tau = \tau(\hat{\epsilon})$$

According to (1.3) the strain energy  $U^\epsilon$  is then given by the volume integral of (1.27). Note however that it is very difficult to express  $U^\epsilon$  in terms of strains since this requires the solution of (1.26) for  $\tau$  in terms of  $\hat{\epsilon}$ . In general it is not possible to do this analytically. This places a severe

limitation on the use of the principle of minimum potential energy or of minimum strain-energy with Ramberg-Osgood stress-strain relations.

Next we consider the principle of minimum complementary energy for axial shear. Since there are only shear stresses  $\tau_2, \tau_3$  and shear strains  $\epsilon_2, \epsilon_3$  the integrand in  $W^\sigma$ , (1.13), is given by

$$\epsilon_{ij} d\sigma_{ij} = 2(\epsilon_2 d\tau_2 + \epsilon_3 d\tau_3) \quad (1.28)$$

It follows from (1.22-23) that (1.28) is given by

$$\epsilon_{ij} d\sigma_{ij} = \frac{\tau}{G} \left[ 1 + \left( \frac{\tau}{\tau_y} \right)^{n-1} \right] d\tau$$

Integration of this expression from 0 to  $\tau$  yields

$$W^\sigma = \frac{\tau^2}{2G} \left[ 1 + \frac{2}{n+1} \left( \frac{\tau}{\tau_y} \right)^{n-1} \right] \quad (1.29)$$

Expression (1.29) now enters as the integral into the volume integral of  $U_C$ , (1.14).

We now examine the meaning of an admissible stress field  $\tilde{\tau}_2, \tilde{\tau}_3$  in the present case. The only surviving equilibrium equation is

$$\frac{\partial \tilde{\tau}_2}{\partial x_2} + \frac{\partial \tilde{\tau}_3}{\partial x_3} = 0 \quad (1.30)$$

The traction components are

$$\begin{aligned} \tilde{T}_1 &= \tilde{\tau}_2 n_2 + \tilde{\tau}_3 n_3 \\ \tilde{T}_2 &= \tilde{\tau}_2 n_1 \\ \tilde{T}_3 &= \tilde{\tau}_3 n_1 \end{aligned} \quad (1.31)$$

We shall be concerned with cylindrical boundaries in fiber reinforced materials whose generator is in  $x_1$  direction. On such a surface  $n_1=0$ . Therefore the only surviving traction component on such a surface is

$$\tilde{T}_1 = \tilde{\tau}_n = \tilde{\tau}_2 n_2 + \tilde{\tau}_3 n_3 \quad (1.32)$$

Consequently an admissible stress system  $\tilde{\tau}_3, \tilde{\tau}_3$  must satisfy (1.30) and the value  $\tau_n^\circ$  of  $\tilde{\tau}_n$  wherever prescribed on the boundary.

The complementary energy functional (1.16) assumes the form

$$\tilde{U}_C = \int_V \tilde{W}^\sigma dV - \int_{S_u} \tilde{T}_1 u_1^\circ dS \quad (a)$$

$$\tilde{W}^\sigma = \frac{\tilde{\tau}^2}{2G} \left[ 1 + \frac{2}{n+1} \left( \frac{\tilde{\tau}}{\tau_y} \right)^{n-1} \right] \quad (b)$$

(1.33)

$$\tilde{\tau} = \sqrt{\tilde{\tau}_2^2 + \tilde{\tau}_3^2} \quad (c)$$

## 2. LOWER BOUND FOR AXIAL SHEAR MODULUS

Consider a uniaxially reinforced lamina which is subjected to axial shear  $\tau_0$  in the 1-2 plane on its boundary, fig. 5.

By the average stress theorem, of Ref. 5,

$$\bar{\sigma}_{12} = \tau_0 \quad (2.1)$$

and all other average stresses vanish.

By the average theorem of virtual work, of Ref. 5,

$$\int_V \epsilon_{ij} d\sigma_{ij} = \bar{\epsilon}_{ij} d\sigma_{ij} \quad (2.2)$$

Since the only nonvanishing average stress in the present case is (2.1) we have

$$\bar{\epsilon}_{ij} d\bar{\sigma}_{ij} = 2\bar{\epsilon}_{12} d\tau_0 \quad (2.3)$$

The complementary energy of the body is given by (14) of Appendix A. The surface integral vanishes however in the present case since no displacements are prescribed on the boundary. Now

$$\begin{aligned} U_C &= \int_V W^\sigma dV = \int_V \int_{\sigma=0}^{\sigma} \bar{\epsilon}_{ij} d\bar{\sigma}_{ij} dV \\ &= \int_{\sigma=0}^{\sigma} \int_V \bar{\epsilon}_{ij} d\bar{\sigma}_{ij} dV \int_0^{\tau_0} \bar{\epsilon}_{12} d\tau_0 \end{aligned} \quad (2.4)$$

The last equality following from (2.2, 3).

By definition the effective secant modulus  $G_A^S$  is given by

$$\bar{\epsilon}_{12} = \frac{\bar{\sigma}_{12}}{G_A^S(\bar{\sigma}_{12})} = \frac{\tau_0}{2G_A^S(\tau_0)} \quad (2.5)$$

Hence (2.4) assumes the form

$$U_C = V \int_0^{\tau_0} \frac{\tau_0 d\tau_0}{G_A^S(\tau_0)} \quad (2.6)$$

In order to find a bound on  $G_A^S$  it will be necessary to find a bound on (2.6) by use of the principle of minimum complementary energy.

It is assumed that the fibers are infinitely rigid in comparison to the matrix. Therefore at fiber/matrix interface

$$u_1 = 0 \quad (2.7)$$

and the only contribution to the complementary energy is from the matrix. Thus, the surface integral in (1.33a) vanishes and it can be written as

$$\tilde{U}_C = \int_{V_m} \tilde{W}^\sigma dV \quad (2.8)$$

where  $V_m$  is the matrix volume.

Furthermore, by (2.3.3) the actual stresses are functions of  $x_2, x_3$  only. It is therefore natural to also choose admissible stresses as functions of  $x_2, x_3$ . Thus  $\tilde{W}^\sigma$  in (1.33) becomes a function of  $x_2, x_3$  only and therefore without loss of generality (1.33a) can be taken over unit length in fiber direction. Thus it can be written

$$\tilde{U}_C = \int_{A_m} \tilde{W}^\sigma(x_2, x_3) dx_2 dx_3 \quad (2.9)$$

In order to construct an admissible stress system it is necessary to devise a geometrical model for a uniaxially reinforced material. In past analyses of FRM two kinds of models have been successfully treated: Periodic arrays of identical circular fibers have been analyzed numerically with the aid of computers and the composite cylinder assemblage model has been treated analytically [1,5] yielding simple closed results. Since the present treatment is to be analytical the composite cylinder assemblage model will be used. A detailed description of the model has been given in [5]. Suffice it to say here that the model represents a cylindrical specimen of a fiber reinforced material as an assemblage of composite cylinders of different sizes which fill the space in the limit. In each composite cylinder the inner cylinder is a fiber and the outer shell is matrix material.



In all cylinders the ratios of fiber to matrix shell radius are the same, (figure 26).

It is recalled that an admissible stress system must satisfy equilibrium and boundary conditions. An obvious possibility for such an admissible field are the stresses of the elastic solution since they certainly satisfy the required conditions. These stresses are the same in any composite cylinder of the assemblage and are given in cylindrical coordinates by (see [5])

$$\tilde{\sigma}_{rz} = \tilde{\tau}_r = \frac{\tau_o}{1+c} \left(1 + \frac{a^2}{r^2}\right) \cos \theta \quad (2.10)$$

$\tilde{\sigma}_{\theta z} = \tilde{\tau}_\theta = -\frac{\tau_o}{1+c} \left(1 - \frac{a^2}{r^2}\right) \sin \theta$   
where  $c$  is the volume fraction of fibers,  $a$  is the radius of any fiber and  $r, \theta$  are polar coordinates, fig. 26.

Since  $\tilde{\tau}$  as expressed by (1.33c) is an invariant with respect to rotations about  $x, = z$  we have also

$$\tilde{\tau}^2 = \tilde{\tau}_r^2 + \tilde{\tau}_\theta^2 \quad (2.11)$$

Substituting (2.10) into (2.11) yields

$$\tilde{\tau}^2 = p^2 \left(1 + \frac{1}{\rho^4} + \frac{2}{\rho^2} \cos \theta\right) \quad (2.12)$$

where

$$p = \frac{\tau_o}{1+c} \quad \rho = \frac{r}{a} \quad (2.13)$$

To simplify the analysis the exponent  $n$  in (1.22) will be assigned the value

$$n = 3 \quad (2.14)$$

It has been found that with this value of  $n$ , experimentally obtained shear stress-strain relations of epoxy can be quite accurately represented with proper choice of  $\tau_y$ . Recalling (1.33), (2.9) then assumes the form

$$\tilde{U}_C = \frac{1}{2G} \int_{A_m} \tilde{\tau}^2 \left[1 + \frac{1}{2} \left(\frac{\tilde{\tau}^2}{y}\right)^2\right] dA \quad (2.15)$$

where  $G$  is the matrix elastic shear modulus. Let the assemblage consist of  $K$  composite cylinder. Define  $\tilde{U}_C^k$  for the  $k$ th composite cylinder by

$$\tilde{U}_C^k = \frac{1}{2G} \int_{A_{mk}} \tilde{\tau}^2 \left[ 1 + \frac{1}{2} \left( \frac{\tilde{\tau}}{\tau_Y} \right)^2 \right] dA \quad (2.16)$$

where  $A_{mk}$  is the matrix area  $a_k \leq r \leq b_k$  in the  $k$ th composite cylinder. Then

$$\tilde{U}_C = \sum_{k=1}^K \tilde{U}_C^k \quad (2.17)$$

Since  $\tilde{\tau}^2$  has been expressed in polar coordinates, (2.12), it is convenient to also evaluate (2.16) in the same coordinates. Using the variable  $\rho$  we have

$$\tilde{U}_C^k = \frac{1}{2G} \int_1^\beta \int_0^{2\pi} \tilde{\tau}^2 \left[ 1 + \left( \frac{\tilde{\tau}}{\tau_Y} \right)^2 \right] \rho d\rho d\theta \quad (2.18)$$

where

$$\beta = b_k/a_k \quad (2.19)$$

which by construction has the same value in all composite cylinders. Note also that the volume fraction of fibers  $c$  is given by

$$c = \left( \frac{a_k}{b_k} \right)^2 = \frac{1}{\beta^2} \quad (2.20)$$

Substituting (2.12) into (2.18) and carrying out the integration we have

$$\tilde{U}_C^k = \frac{\pi b_k^2}{2G} \tau_o^2 \left[ \frac{1-c}{1+c} + \left( \frac{\tau_o}{\tau_Y} \right)^2 \frac{3+10c-12c^2-c^4}{6(1+c)^4} \right] \quad (2.21)$$

where (2.20) has been used. It is seen that  $\pi b_k^2$  is the area of the cross section of the kth composite cylinder and the parenthesis has the same value for all composite cylinders. Therefore, if (2.21) is inserted into (2.17) we find

$$\tilde{U}_C = \frac{A}{2G} \tau_o^2 \left[ \frac{1-c}{1+c} + \left( \frac{\tau_o}{\tau_Y} \right)^2 \frac{3+10c-12c^2-c^4}{6(1+c)^4} \right] \quad (2.22)$$

Let (2.22) be written

$$\tilde{U}_C = A \int_0^{\tau_o} \frac{1}{A} \frac{d\tilde{U}_C}{d\tau_o} d\tau_o \quad (2.23)$$

Without loss of generality (2.6) can be evaluated for unit height of cylindrical specimen. Thus

$$U_C = A \int_0^{\tau_o} \frac{\tau_o d\tau_o}{G_A^S(\tau_o)} \quad (2.24)$$

Now introduce (2.23) and (2.24) into the minimum complementary inequality (1.18). Thus

$$\int_0^{\tau_o} \left[ \frac{1}{A} \frac{d\tilde{U}_C}{d\tau_o} - \frac{\tau_o}{G_A^S(\tau_o)} \right] d\tau_o \geq 0 \quad (2.25)$$

Since the integral is positive for all values of  $\tau_o$ , the integrand must also be positive for all values of  $\tau_o$ . It follows that

$$G_A^S(\tau_o) \geq \frac{A\tau_o}{dU_C/d\tau_o} = G_A^S(-) \quad (2.26)$$

where the extreme right denotes lower bound on the secant modulus  $G_A^S$ . Substituting (2.22) into (2.26) and rearranging we find the lower bound (2.3.9) of Chapter 2.

There naturally arises the question of the establishment of an upper bound. The difficulties involved have been discussed above: It is not in general possible to solve Ramberg-Osgood relations for stresses in terms of strains. It is

therefore not possible to analytically express the potential energy functional in terms of admissible strains.

A possibility to resolve the difficulty is to write inelastic stress-strain relations of type (1.22) in the form

$$\begin{aligned}\tau_2 &= 2G\epsilon_2 \left[ 1 - \left( \frac{\hat{\epsilon}}{\epsilon_y} \right)^{\alpha-1} \right] \\ \tau_3 &= 3G\epsilon_3 \left[ 1 - \left( \frac{\hat{\epsilon}}{\epsilon_y} \right)^{\alpha-1} \right] \\ \hat{\epsilon} &= \sqrt{\epsilon_2^2 + \epsilon_3^2}\end{aligned}\tag{2.27}$$

where  $\alpha$  and  $\epsilon_y$  are to be determined by curve fitting. The minus sign in the parenthesis is due to the fact that the stress-strain curve is below a straight line with the initial slope.

It should be noted that (2.27) are not an inversion of (1.22). They are merely another form of approximation of actual stress-strain curves.

In principle the representation (2.27) can now be used in conjunction with the principle of minimum potential energy to establish an upper bound on  $G_A^S$  in same fashion as a lower bound has been established. It has however been found that in attempting to fit (2.27) to actual epoxy stress-strain curves a fractional exponent  $\alpha$  was needed. This led to integrals of formidable difficulty in the evaluation of potential energy functionals. Therefore this approach has not been continued here.

APPENDIX D  
FAILURE OF NON-LINEAR LAMINATES

It is expedient to separate the problem of the establishment of failure criteria of laminates into two separate problems:

- (a) Establishment of failure criteria for uniaxially fiber reinforced material, i.e., laminae.
- (b) Establishment of failure criteria of the laminate on the basis of laminae failure criteria.

A great deal of work has been done on problem (a). The problem has been approached in micro as well as macro-fashion. In micro-approach, it is attempted to predict failure on the basis of local analysis of the interior of the composite. Such an approach evidently encounters extreme difficulties. Although important work of fundamental nature has been done in this area, we shall not be concerned with it here since the work has not advanced to the stage of prediction of failure criteria under states of combined stress.

In the macro-approach, a failure criterion is heuristically postulated as some function of pertinent state variables (generally average stresses) which also contains undetermined parameters. These parameters are then to be determined in terms of experimentally accessible information.

We shall in the present discussion limit ourselves to states of plane stress. The simplest failure criterion is the so-called maximum stress criterion which states that failure occurs when either one of: stress in fiber direction, stress transverse to fibers, shear stress, reaches its critical value, these critical values being the same whether or not the stresses act simultaneously. In symbols the criterion is:

$$\sigma_{11} = \sigma_A$$

or

$$\sigma_{22} = \sigma_T$$

(1)

62.

or

$$\sigma_{12} = \tau_{AT}$$

where 1 is fiber direction and 2 is the transverse direction.

Generally, failure stresses  $\sigma_A$  and  $\sigma_T$  are different in tension and compression. This is known as Bauschinger effect. There is evidently no Bauschinger effect for the shear stress. The simplest generalization of (1) to account for Bauschinger effect would be to assume as failure criterion:

$$\begin{aligned}
 \sigma_{11} &= \sigma_A^+ & \text{if} & & \sigma_{11} > 0 \\
 \sigma_{11} &= \sigma_A^- & \text{if} & & \sigma_{11} < 0 \\
 \sigma_{22} &= \sigma_T^+ & \text{if} & & \sigma_{22} > 0 \\
 \sigma_{22} &= \sigma_T^- & \text{if} & & \sigma_{22} < 0 \\
 \sigma_{12} &= \tau_{AT} & \text{all} & & \sigma_{12}
 \end{aligned} \tag{2}$$

whichever occurs first, where (+) and (-) superscripts denote failure stresses in tension and compression respectively. The main drawback of these simple criteria is in that they take no account of interaction effects.

The most commonly used criterion which takes into account interaction is of quadratic form. For plane stress it has the form

$$A_{11}\sigma_{11}^2 + A_{22}\sigma_{22}^2 + A_{12}\sigma_{11}\sigma_{22} + A_{44}\sigma_{12}^2 = 1 \tag{3}$$

Here, products of shear stress with normal stress have been omitted since the material cannot distinguish between positive and negative shear stress. Therefore, odd powers (one, in this case) of shear stress cannot appear.

Applying (3) to failure for stress in fiber direction alone, stress transverse to fiber direction alone, shear stress alone, in turn, it is seen at once that

$$\begin{aligned}
A_{11} &= \frac{1}{\sigma_A^2} \\
A_{22} &= \frac{1}{\sigma_T^2} \\
A_{44} &= \frac{1}{\tau_{AT}^2}
\end{aligned}
\tag{4}$$

The coefficient  $A_{12}$  is troublesome since its determination requires a failure experiment under combined stress. Several authors have proposed to use failure experiments on off-axis specimens under uniaxial stress for the determination of  $A_{12}$ . See e.g. [18] for discussion.

The situation becomes more complicated if it is required to take into account Bauschinger effect, that is difference of failure stresses in tension and compression. One possibility to account for this effect is to assume that  $A_{11}$ ,  $A_{22}$  assume different values for tension and compression. The situation regarding  $A_{12}$ , however, becomes very awkward as it would have to assume four different values to account for four different possibilities of sign combination in biaxial stressing and

It is also possible to add linear terms to (3) in which case it would assume the form:

$$\begin{aligned}
&A_{11}\sigma_{11}^2 + A_{22}\sigma_{22}^2 + A_{12}\sigma_{11}\sigma_{22} + A_{44}\sigma_{12}^2 + \\
&B_1\sigma_{11} + B_2\sigma_{22} = 1
\end{aligned}
\tag{5}$$

Such a device was suggested by Hoffman [19]. In this case it is possible to determine values of  $A_{11}$ ,  $B_1$ ,  $A_{22}$ ,  $B_2$  to account for different tensile and compressive uniaxial failure stresses in fiber direction and transverse to it. But the difficulty of assigning four different values to  $A_{12}$  remains, unfortunately.

In summary, the status of quadratic failure criteria has to date not been finalized. However, special versions of such criteria have been successfully fitted to experimental data.

It is of importance to realize that in the fiber reinforced materials used in practice failure predictions on the

basis of maximum stress criterion or quadratic failure criterion are not very different. This is due to the large ratios between strength in fiber direction and transverse and shear strengths and is easiest realized by considering the failure criteria as surfaces in  $\sigma_{11}$ ,  $\sigma_{22}$ ,  $\sigma_{12}$  stress space. The maximum stress criterion is a very elongated rectangular parallelepiped while the quadratic failure criterion is an ellipsoid. For  $A_{12}=0$ , Fig. 25 shows this schematically on a cut in the  $\sigma_{11}$ ,  $\sigma_{22}$  plane. Thus it is seen that stress points on the two failure surfaces are close together for most parts of the surfaces.

The situation would be entirely different for a material in which  $\sigma_A$ , and  $\sigma_T$  were of comparable magnitudes.

We shall now consider problem (b) i.e., the establishment of laminate failure criteria in terms of laminae failure criteria. The most conservative laminate failure criterion is to assume that once any lamina has failed the laminate has reached its ultimate load. There are cases of laminates in which all laminae would fail simultaneously and then this criterion would be justified. For example: a  $\pm\theta$  laminate in which the external load direction bisects the angle between the fibers.

In most cases, however, a certain group of laminae will fail first and failure of remaining groups would require further increase of load. Therefore a more realistic alternative is to determine the load at which the first laminae group fails. At this state, the further carrying capacity of the laminate may be assumed to be given by the remaining undamaged laminae. The increase in load which fails another group of laminae is then determined. This process is continued until failure of all laminae has taken place.

Still another possibility is to assume that when a lamina has failed, certain of its stiffnesses reduce to zero. For example: suppose that a lamina or group of laminae has failed in shear. Such a failure implies a crack through the lamina in fiber direction. In that event, it is reasonable to assume



that the shear and transverse stiffnesses of the lamina are zero, but it still retains its stiffness in fiber direction. If, however, a lamina fails because of the stress in fiber direction the damage is so widespread that all of its stiffnesses will be negligible. According to the type of failures encountered analysis is continued for the damaged laminate with the new stiffness rearrangement. This process is continued until failure of all laminae has taken place. This method of analysis seems to be the most realistic but is also the most complicated.

In almost all of the practical strength analyses of laminates in the literature, according to any of the methods outlined above, the stresses used for failure criteria have been determined on the basis of elastic laminate analysis. With the present inelastic laminate analysis, more realistic stresses are available in a better assessment of laminate failure loads.

## APPENDIX E

### MSC-NOLIN COMPUTER PROGRAM

#### 1. General Description of the Program

This is a computer program developed for the inelastic analysis of a laminate subject to any constant, arbitrary combination of in-plane loading. Details of the method of analysis and of the numerical solution, using the Newton-Raphson method, have been described in the body of this report. The essential features of the program are summarized below.

The primary capability of MSC-NOLIN is to compute laminae properties when the laminate loads are defined. There is also a limited capability to work with constituent properties, rather than laminae properties, as the input. Details of the input options are discussed subsequently. Basically, the inputs required are the stress-strain characteristics of the individual laminae for each of the three in-plane stress components applied separately. The stress-strain curves for transverse stress and for stress and for axial shear stress are defined by Ramberg-Osgood stress-strain curves. The parameters for these curves along with the laminae elastic constants are the required material property inputs.

It has been observed that axial shear stresses in individual laminae are a major, perhaps the major, source of nonlinearities in laminate response. Therefore, several additional options have been included in the MSC-NOLIN to accommodate more detailed characterization of shear response. First, the laminae shear stress-strain response may be input in tabular form and a least squares fit to the data is automatically obtained for the R-O yield stress (limited to the use of an exponent,  $n=3$ ). Secondly, the matrix shear stress-strain curve can be input along with fiber elastic properties and the laminae shear stress-strain curve will be computed. In this latter case, the laminae elastic constants are also computed.

The input specifies one of two options for the determination of the initial set of stresses to be used in the iteration at each value of applied load on the laminate. In one case the stresses found at one load are increased to the load for which the stresses were evaluated. In the other, and generally used option, the increment between the initial stresses used at the nth laminate load value and the actual stresses found for the (n-1)st load value bears the same relation to the ratio of those two load values as the similar relation computed at the previous load cycle, that is,

$$\frac{\sigma_{ij}^{(n)} - \sigma_{ij}^{(n-1)}}{F_n/F_{n-1}} = \frac{\sigma_{ij}^{(n-1)} - \sigma_{ij}^{(n-2)}}{F_{n-1}/F_{n-2}}$$

The program contains a number of controls to define: the size and number of steps of loading at which computations are made; the maximum number of iterations to be permitted in the numerical solution; the desired accuracy to be obtained in convergence; the criteria for divergence of the solution in the iterative process to avoid the use of unnecessary execution time in the case of breakdown of the solution procedure. The program defines the failure of the laminate in a limited fashion, either on the basis of the maximum allowable stress in the fiber in tension or compression, or on the basis that the tangent modulus of the stress-strain curve of the laminate becomes less than a specified value. Failure due to shear or transverse stress are not included at this stage in the development of the program.

## 2. Input

The main features of input in this program are the following:

- (a) Specify the number of laminates or problems to be solved;

- (b) Define the geometrical properties of each layer;
- (c) Define either the material properties of each layer or the properties of its constituents;
- (d) Define either of the following for each layer:
  - (i) yield stress in transverse direction and yield stress in shear;
  - (ii) yield stress in transverse direction and a table of values defining shear stress-strain curve for the matrix plus a set of values of stresses to be used for the computation of yield stress in shear;
- (e) Specify the type of Ramberg-Osgood relation to be used;
- (f) Define the loadings; and
- (g) Define the control parameters.

A guide to the preparation of input data for this program is given in section 4 below.

### 3. Details of Output

The output can be divided basically into two steps:

(a) Output of Input Data:

The first section of the output deals with the output of the input data. If the input is in the form of properties of constituents of the layer, it gives an output of the properties of the constituents first and then the computed value of the properties of the layer; otherwise, it gives output directly the properties of the layer.

(b) Output of Stresses and Strains:

For each set of loading, the computer prints the following:

- (1) value of the load applied;
- (2) number of iterations for convergence;
- (3) stresses for individual laminae with respect to principal elastic axes of the laminae; and
- (4) strains for individual laminae in terms of both laminae and laminate axes.

#### 4. Input Details for MSC-NOLIN

(1) Read (I5) NSETS

NSETS: number of problems

(2) Read (I5) LAY

LAY: number of layers in this laminate analysis

(3) Read (I5) INP

INP: Option for reading in material properties

INP = 1; read in material properties of individual laminae;

INP = 2; compute properties of laminae from the properties of constituents.

(4) (a) If INP = 1

(i) Read (5D15.5)  $E_{11}, E_{22}, \mu_{12}, \mu_{21}$

(ii) Read (5D15.5)  $G_{12}, SY, TY$

(iii) Read (D15.5,I5) T, IANG

(b) If INP = 2

(i) Read (4D15.5) EF, MUF, GF, VF

(ii) Read (3D15.5) EM, MUM, GM

(iii) Read (I5) I2

If I 2 = 0; read in SY and TY  
(i) Read (2D15.5) SY, TY

If I 2 = 1; TY is to be computed

(i) Read (5,1002) SYCE

(ii) Read ( 2 I 5) NUMT

NUMT = number of values in the table  
(iii) Read (5D15.5) TAU (J), J=1, NUMT  
(Table of shear stress values of matrix read in)

(iv) Read (5D15.5) GAM (J), J=1, NUMT  
(Table of shear strain values of matrix read in)

(v) Read (5D15.5) SG12 (J), J=2,11  
(Table of shear stress values of laminae read in)

- (5) Read (5D15.5) XN, XM
- XN: exponent in nonlinear transverse stress-strain law;
- XM: exponent in nonlinear shear stress-strain law.
- (6) Read (5D15.5) S011, S022, S012
- S011: applied stress in X-direction  
S022: applied stress in Y-direction  
S012: shear stress in XY
- (7) Read (I5, D15.5) KSGM, SMLT
- KSGM: total number of loading increments
- SMLT: ratio of load increment to the initial load.
- (8) Read (D15.5) STIFF
- STIFF: tangent modulus of stress-strain curve in terms of the laminate axes; specify a value of STIFF below which the program will not run.
- Read (D15.5) SGR
- SGR: maximum allowable stress in the fiber in tension or compression
- (9) Read (I 5, 2D15.5) IT, EPS, UPBD
- IT: maximum number of iteration permitted in Newton-Raphson analysis
- EPS: convergence criteria; (ratio of values of two successive iterations should be less than EPS)
- UPBD: divergence criteria (solution will stop if ratio of two successive iterations is greater than  $10^{+12}$ )
- (10) Read(I5) INMT
- If INMT = 1, the program uses ratio of previous two solutions as the initial guess value iteration process;
- If INMT = 2, the program uses extrapolated value of previous two solutions proportioned on the basis of stress ratio as the initial guess.

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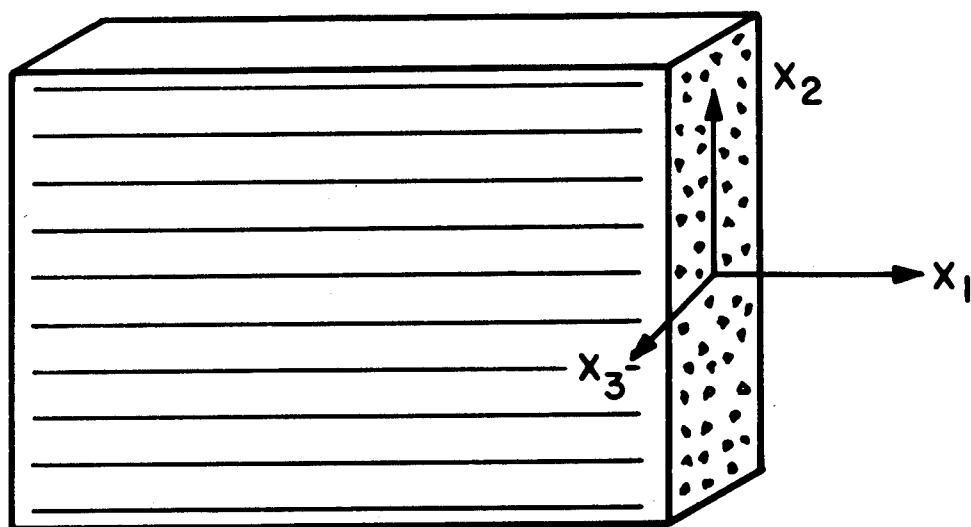


Fig. 1 - Coordinate system for unidirectional fiber composite material.

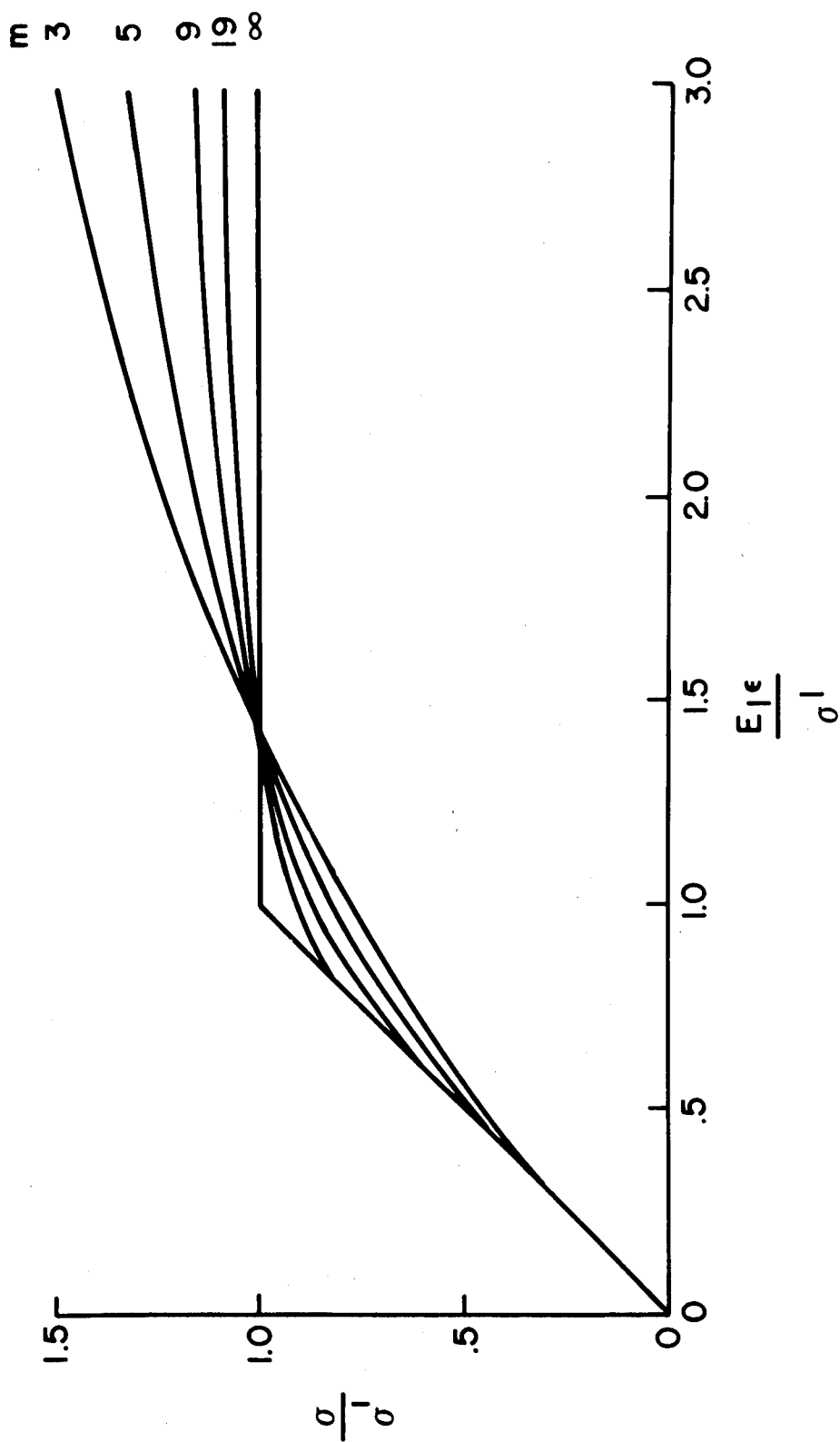


Fig. 2 - Nondimensional Ramberg-Osgood stress-strain curves.  
(e.g., 2.2.1 with  $k = 0.4$ ).

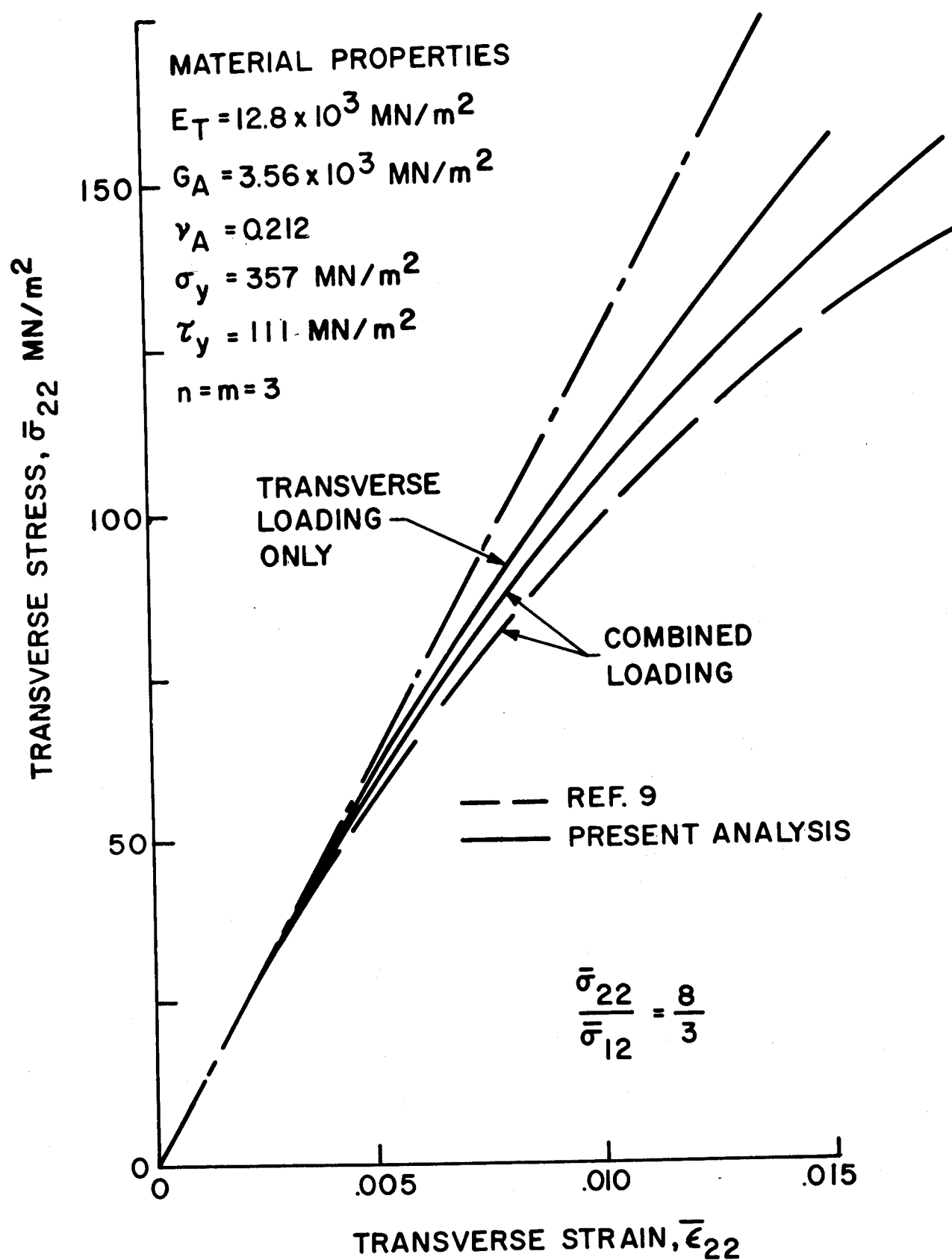


Fig. 3 - Transverse normal  $\sigma$ - $\epsilon$  curves for unidirectional Boron/Epoxy (for normal loading and normal + shear loading).

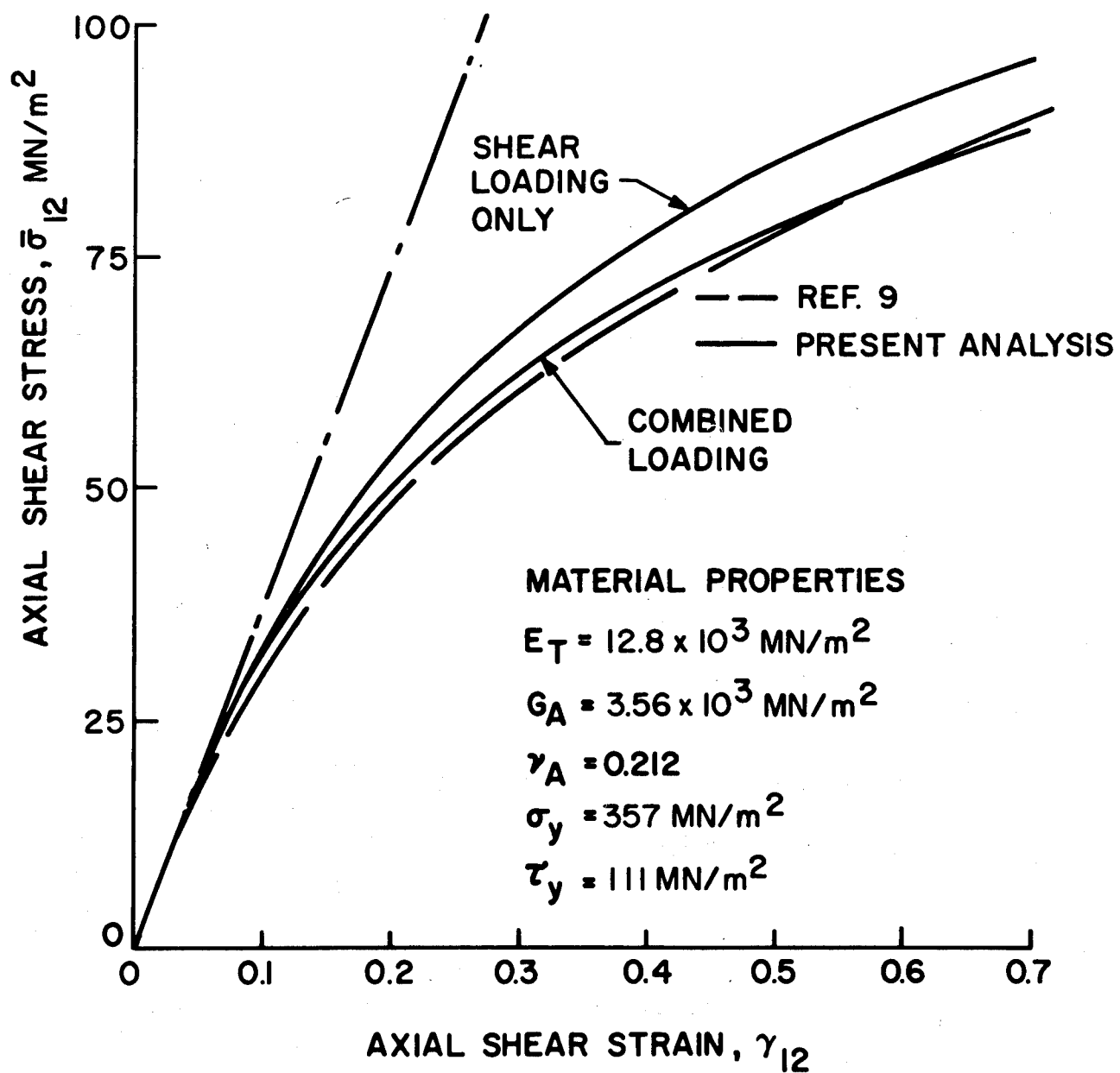


Fig. 4 - Shear  $\sigma$ - $\epsilon$  curves for unidirectional Boron/Epoxy (for shear loading and normal + shear loading).

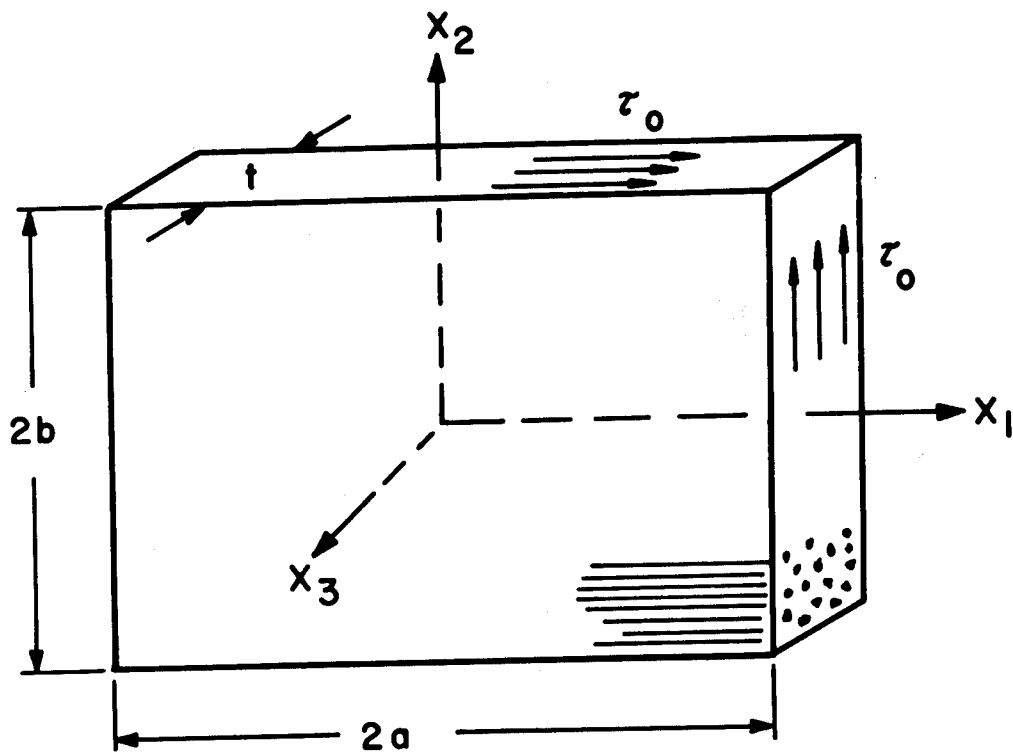


Fig. 5 - Unidirectional fiber composite material under axial shear stress.

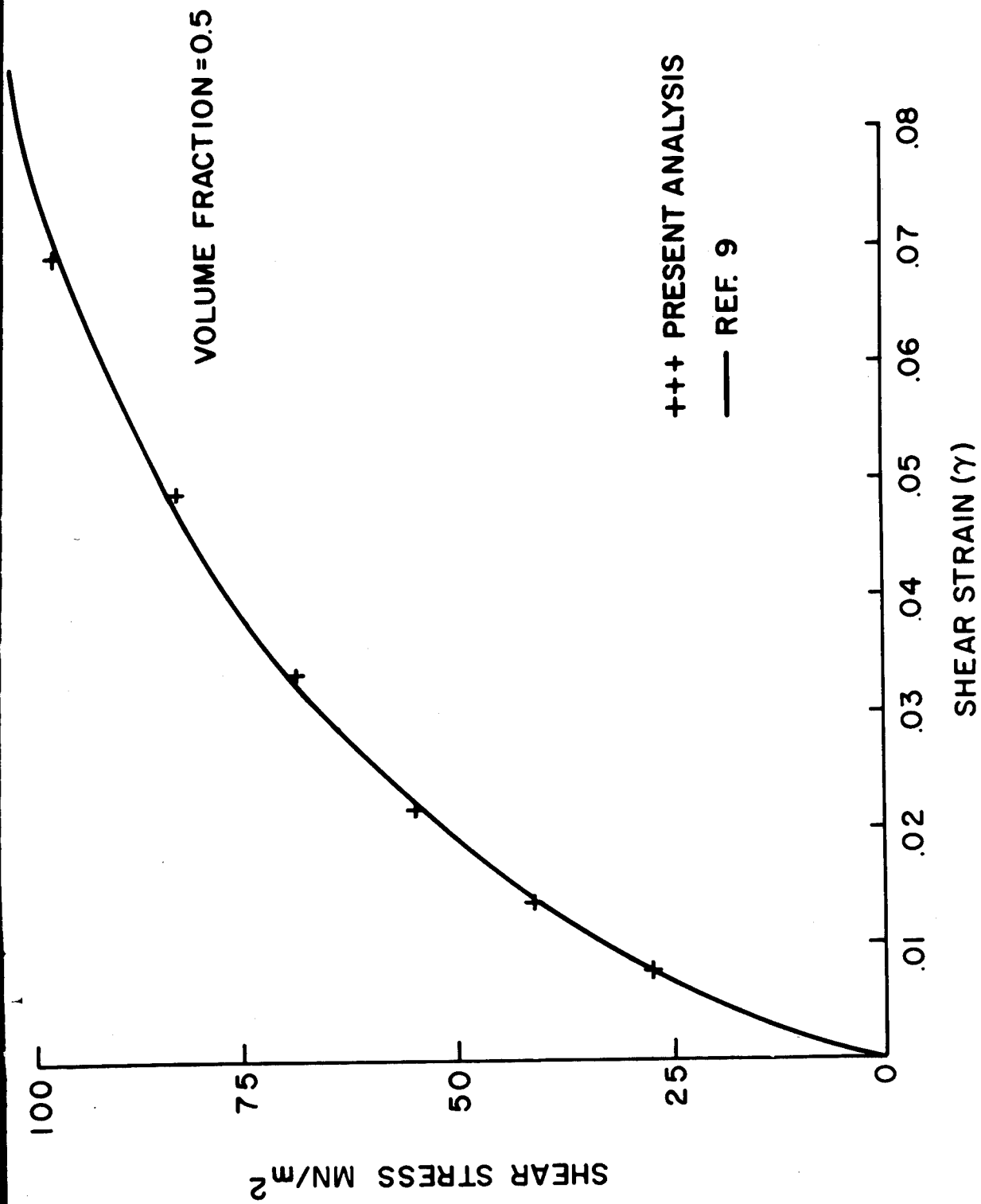


Fig. 6 - Axial shear stress-strain curve for composite computed from matrix properties.

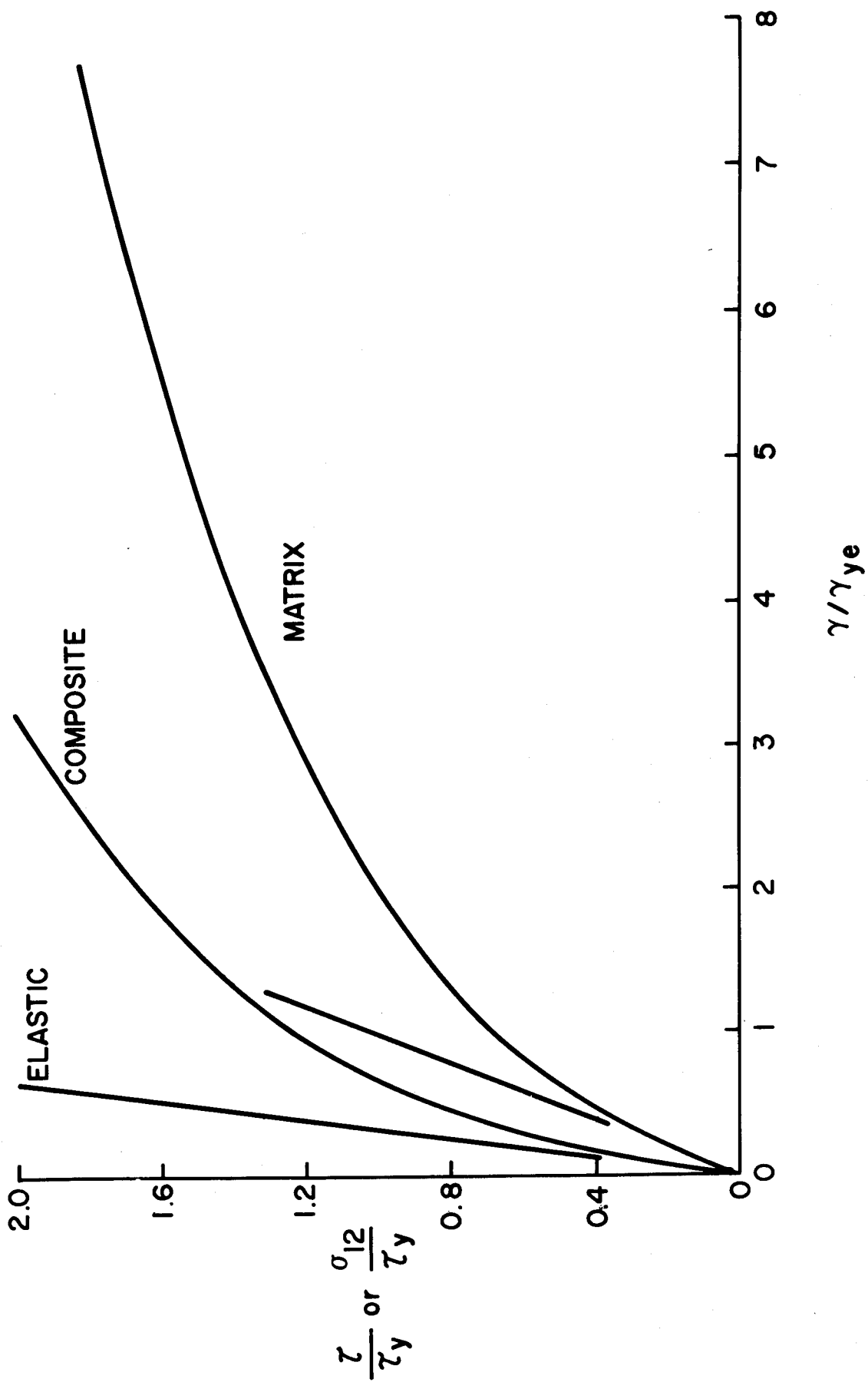


Fig. 7 - Generalized Ramberg-Osgood stress-strain curve in axial shear for composite and matrix ( $n=3$ ).

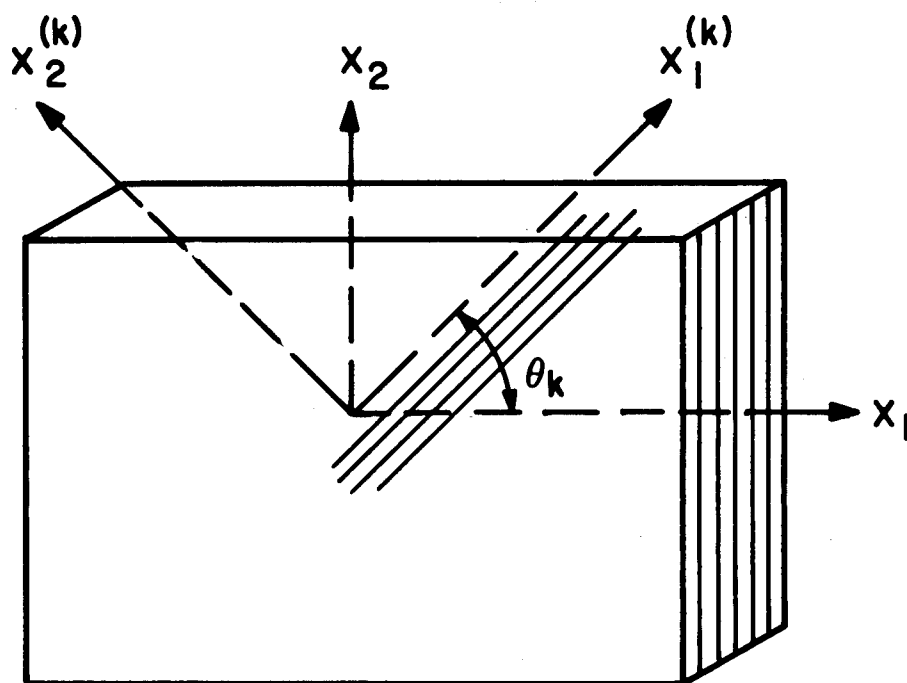


Fig. 8 - Laminate coordinate system.



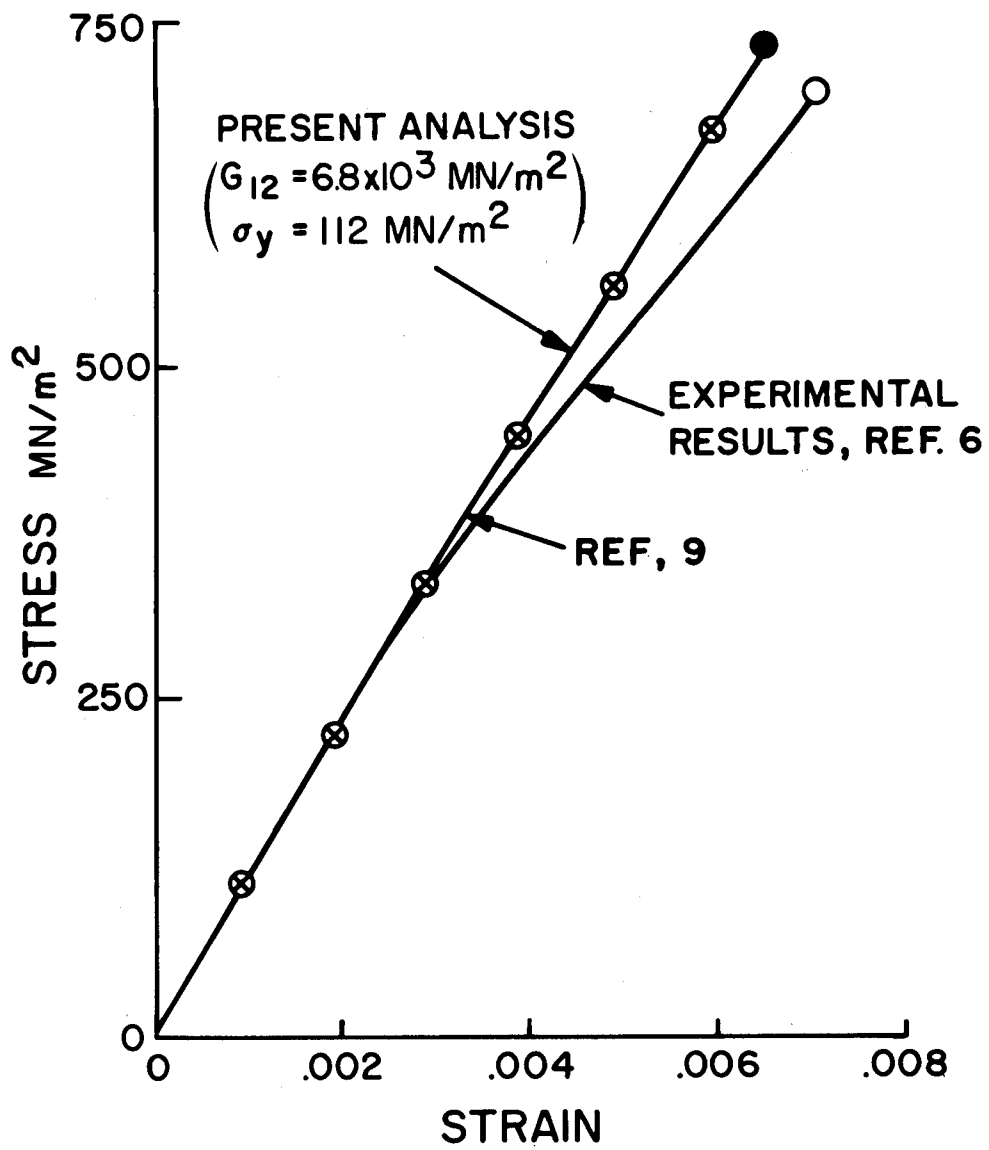


Fig. 9 - Tensile stress-strain curve with [0/90] B/Ep.

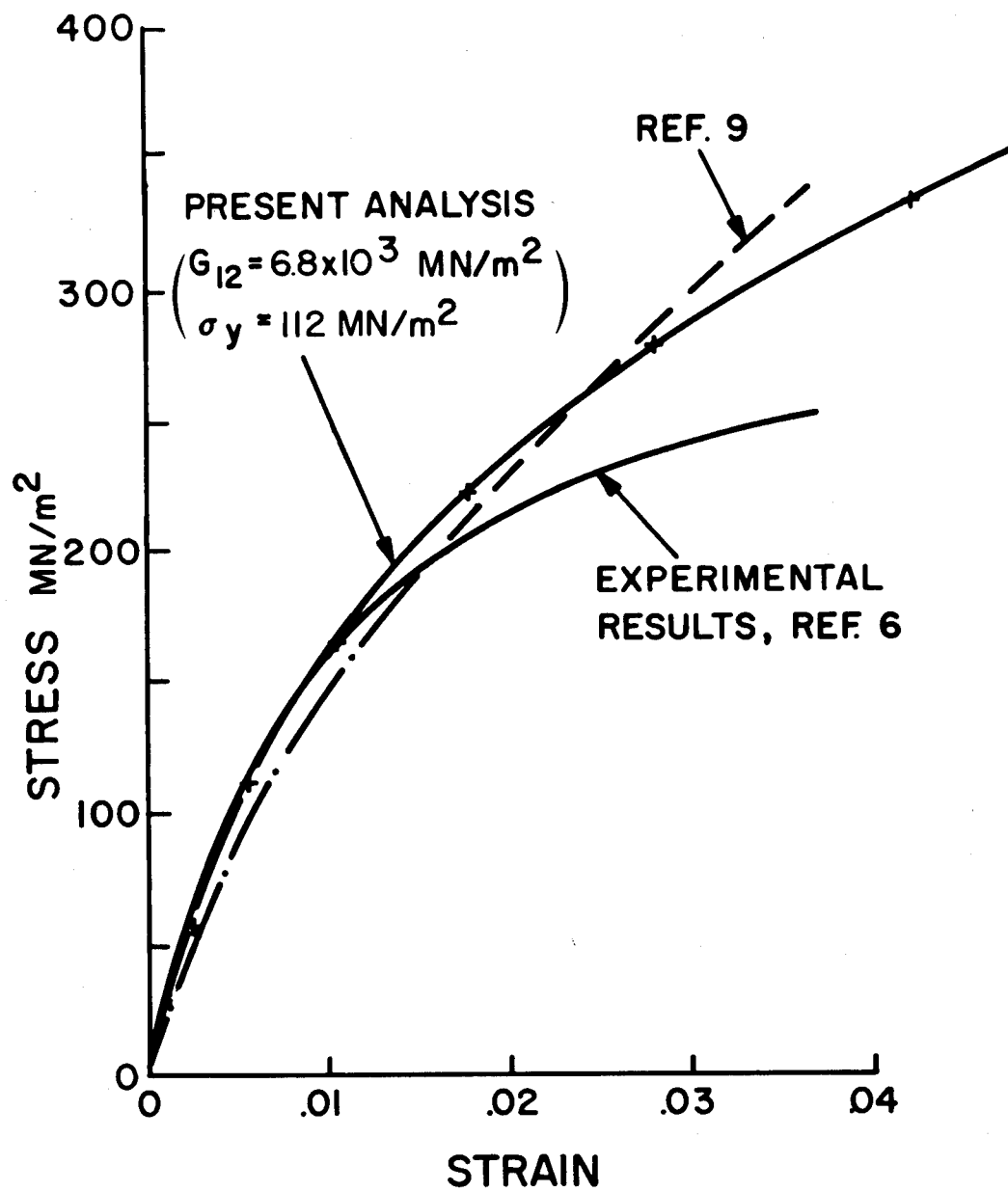


Fig. 10 - Tensile stress-strain curves for  $[\pm 45]$  B/Ep.

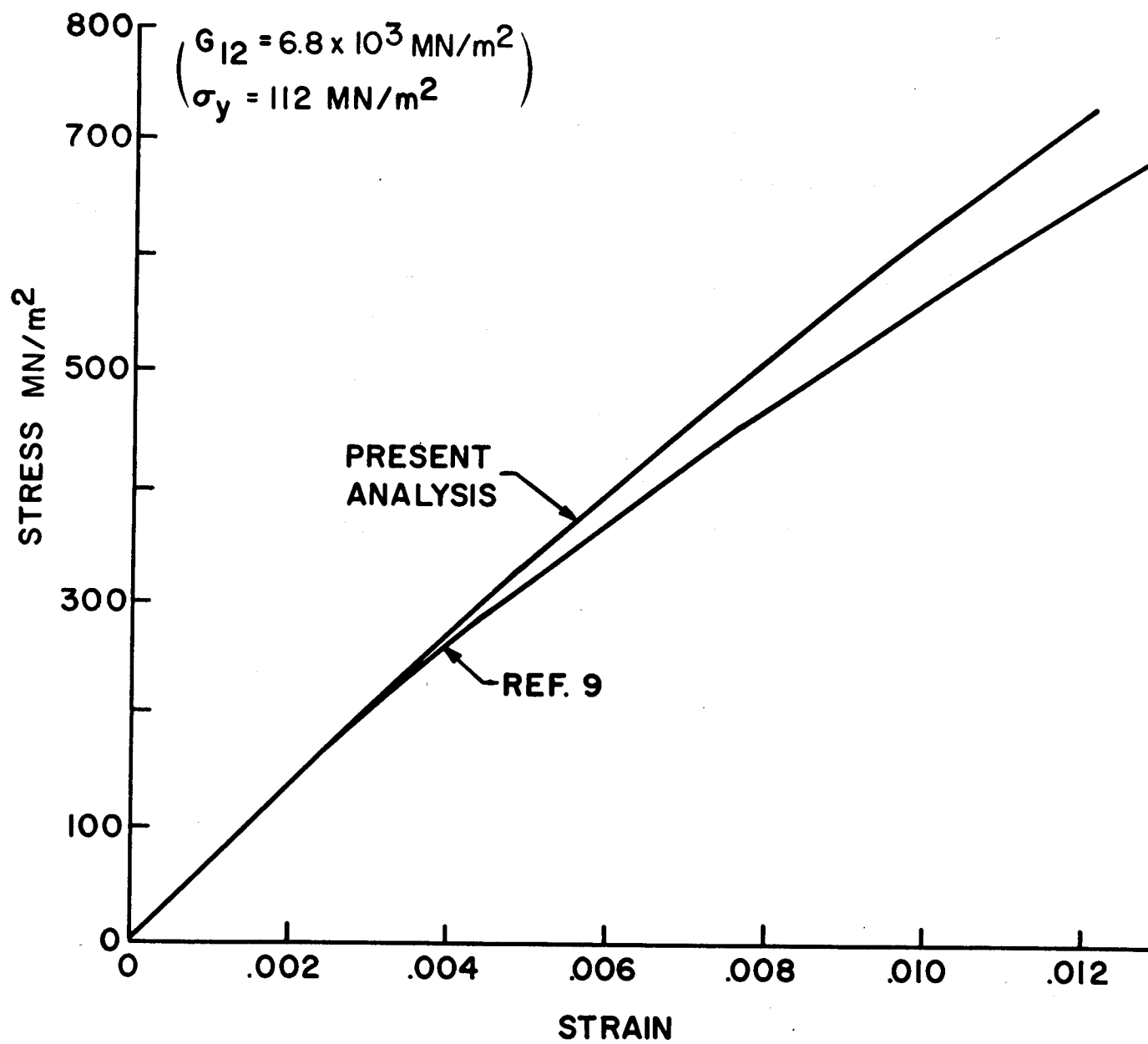


Fig. 11 - Tensile stress-strain curve with [+30] B/Ep.

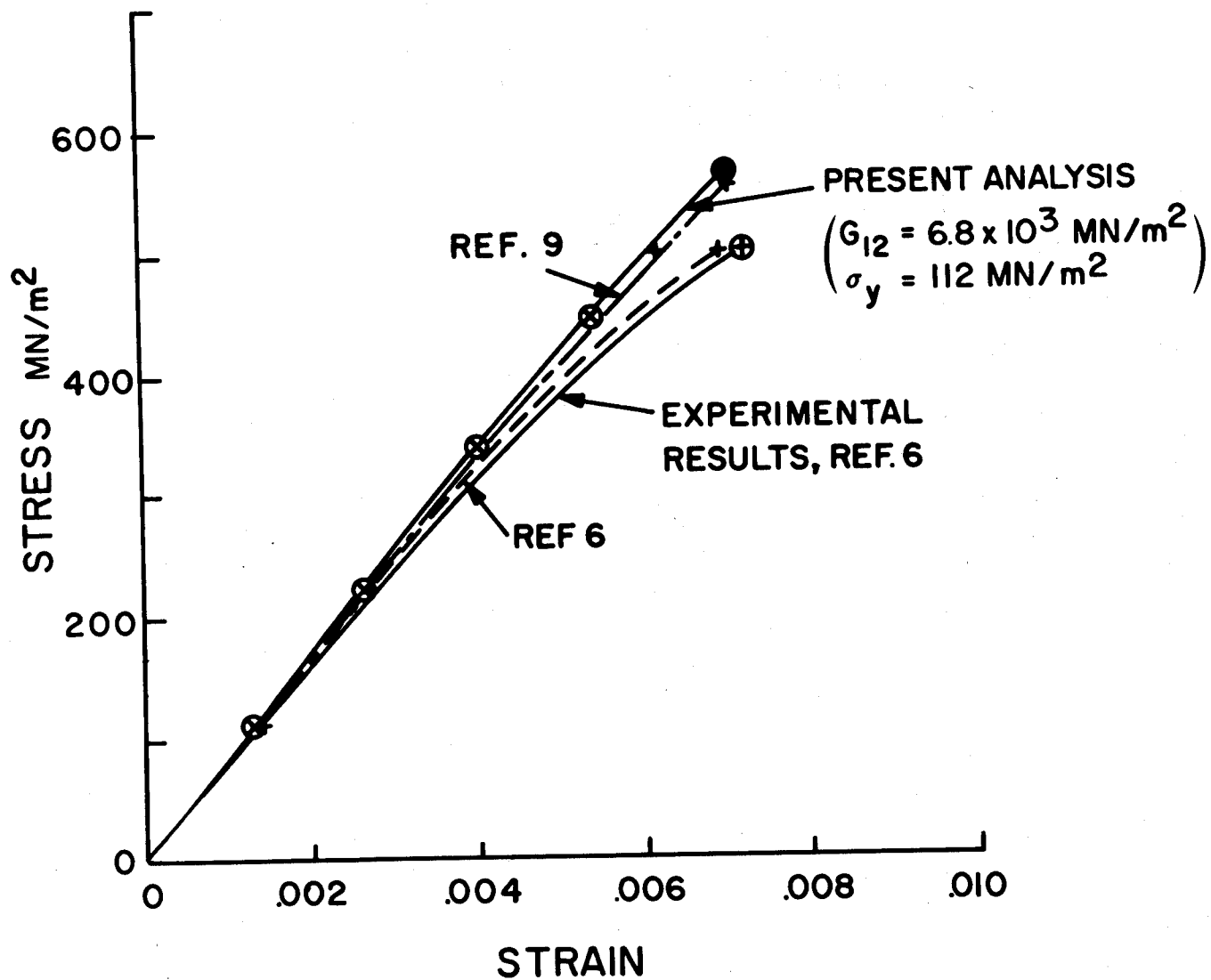


Fig. 12 -  $0^\circ$  Tensile stress-strain curves for  $[0/\pm 45/90]$  B/Ep.

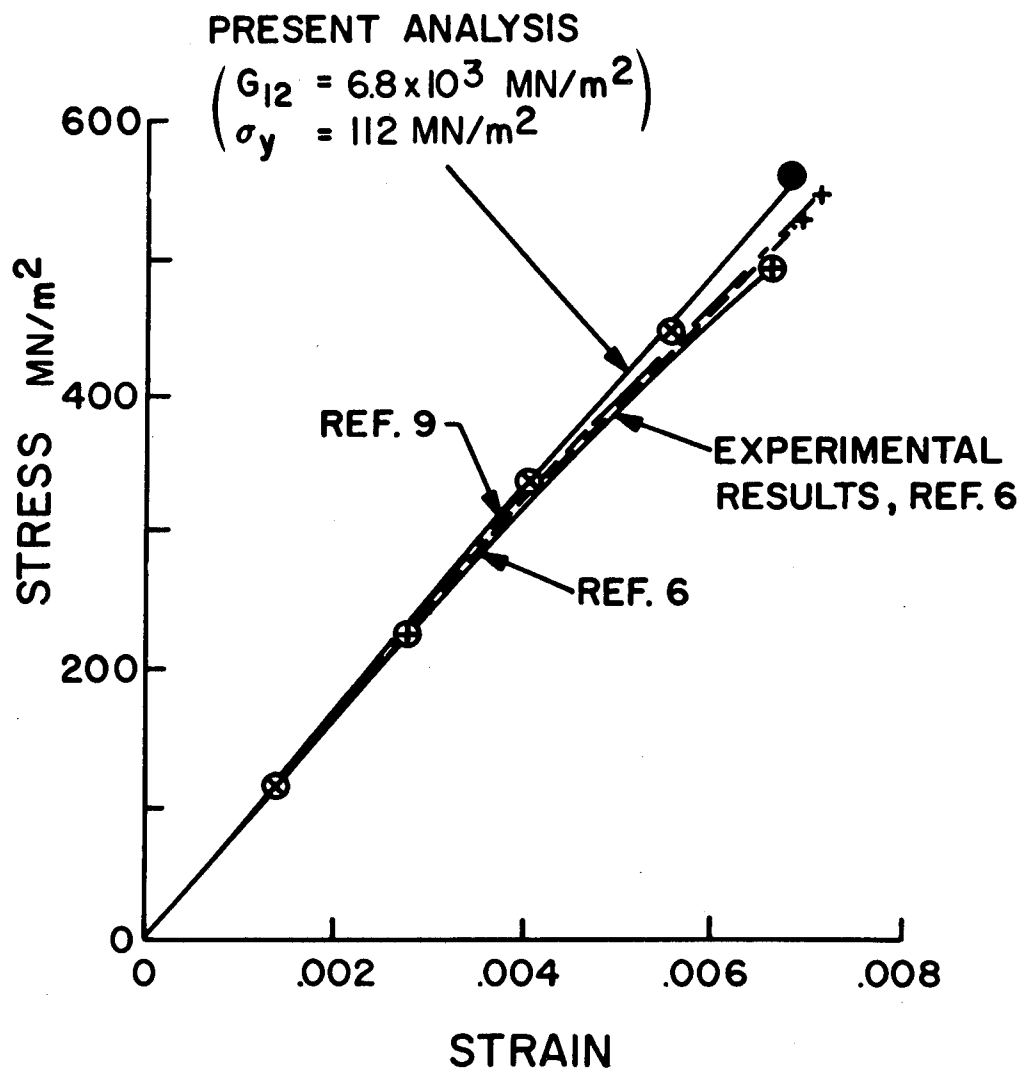


Fig. 13 -  $0^\circ$  Tensile stress-strain curves for  $[0/\pm 60]$  B/Ep.

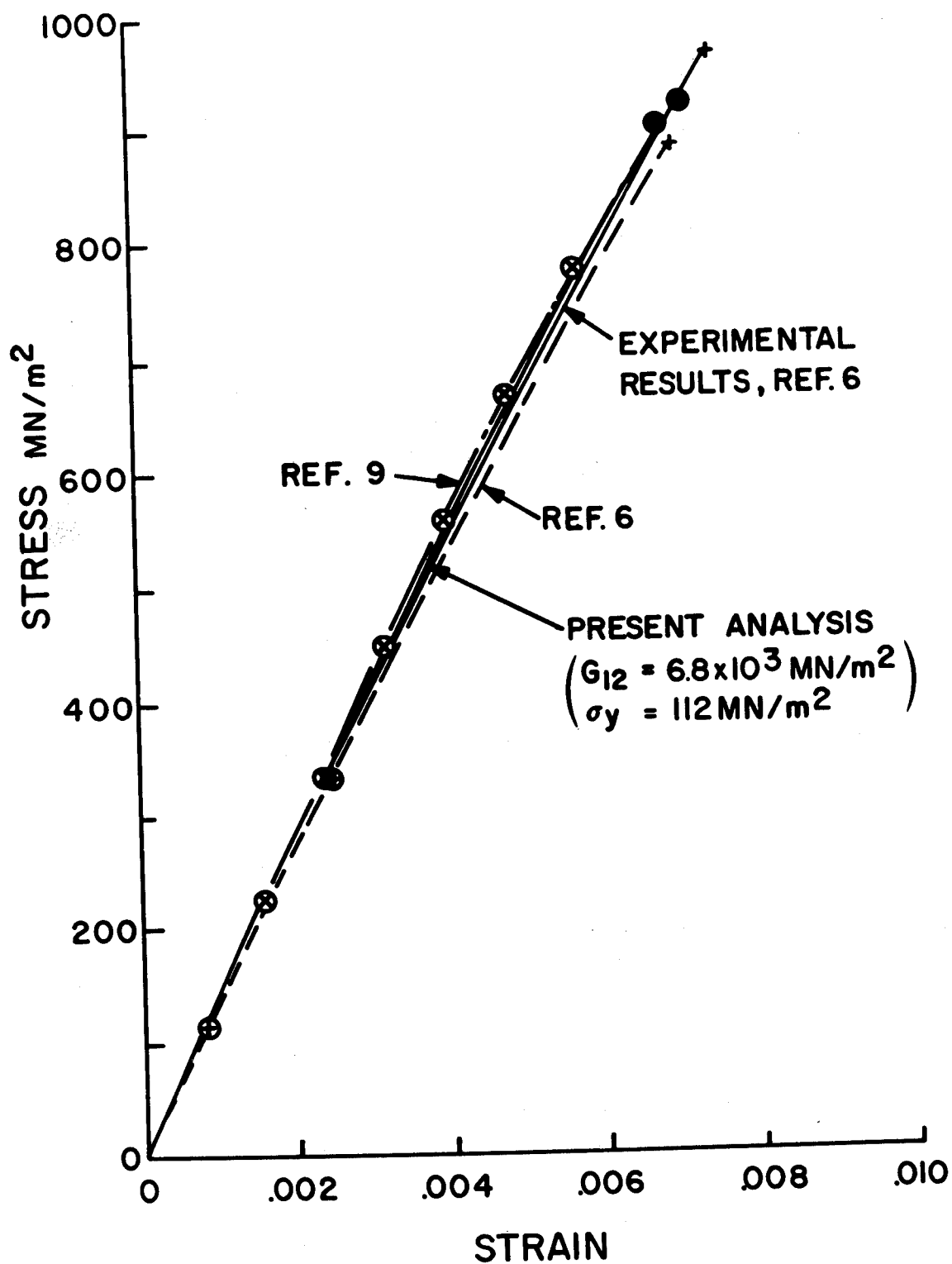


Fig. 14 -  $0^\circ$  Tensile stress-strain curves for  $[0_3/+45]$  B/Ep. 87.

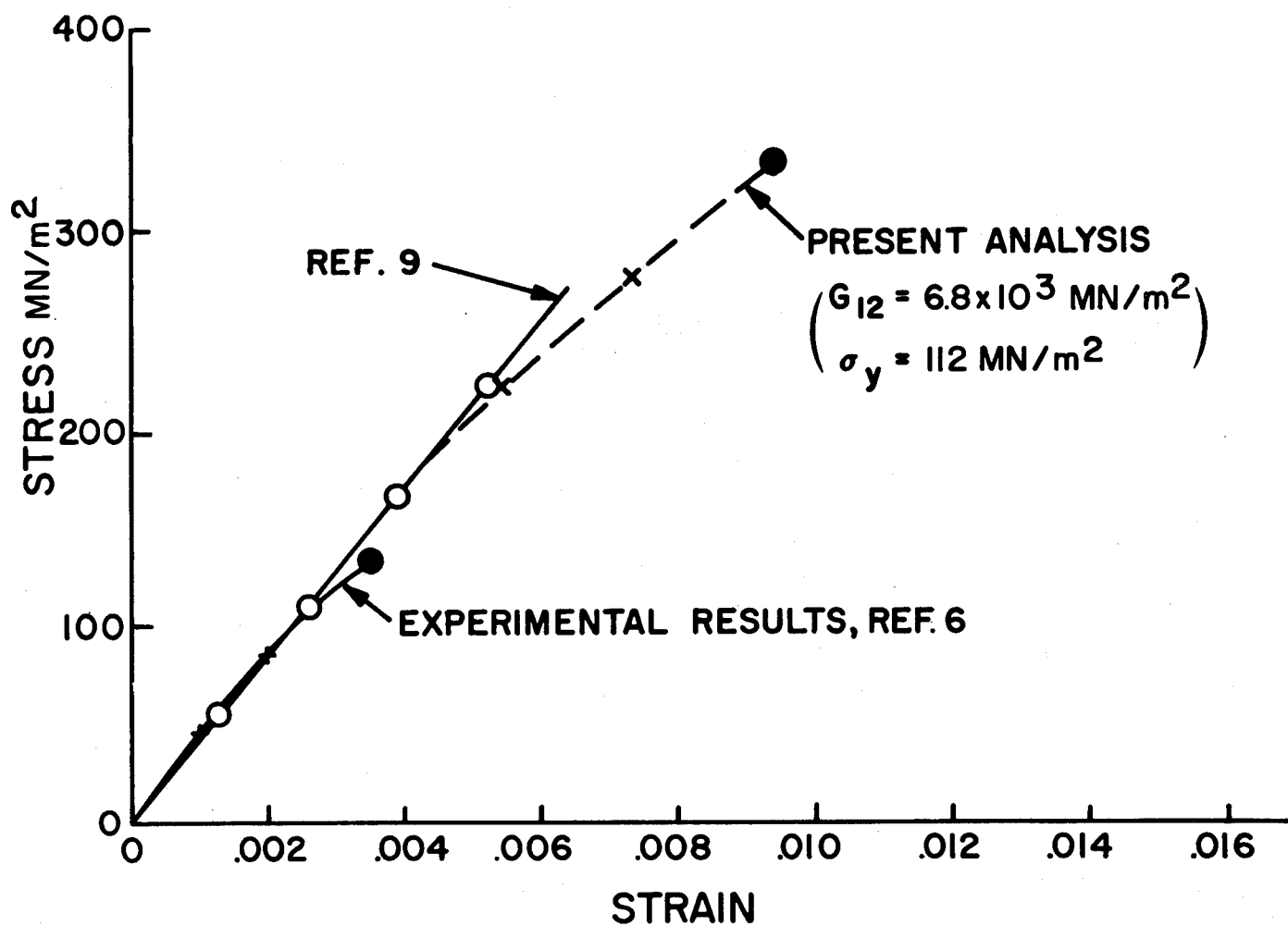


Fig. 15 - 0° Tensile stress-strain behavior of  $[65_3/20/-70]$  B/Ep.

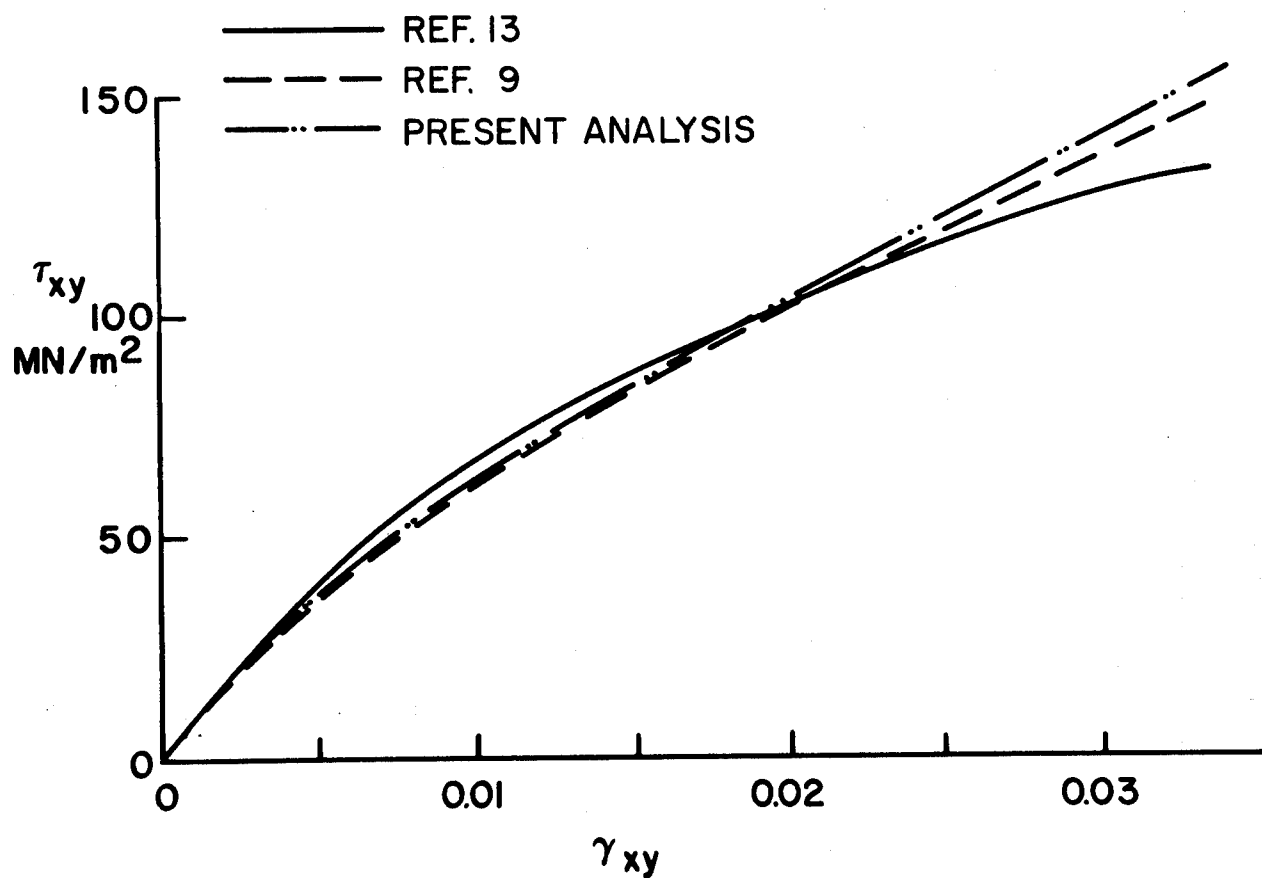


Fig. 16 - Comparison of present results with experimental data for Glass/Epoxy 90/+18 tubes in torsion, and with the results of Ref. 9.



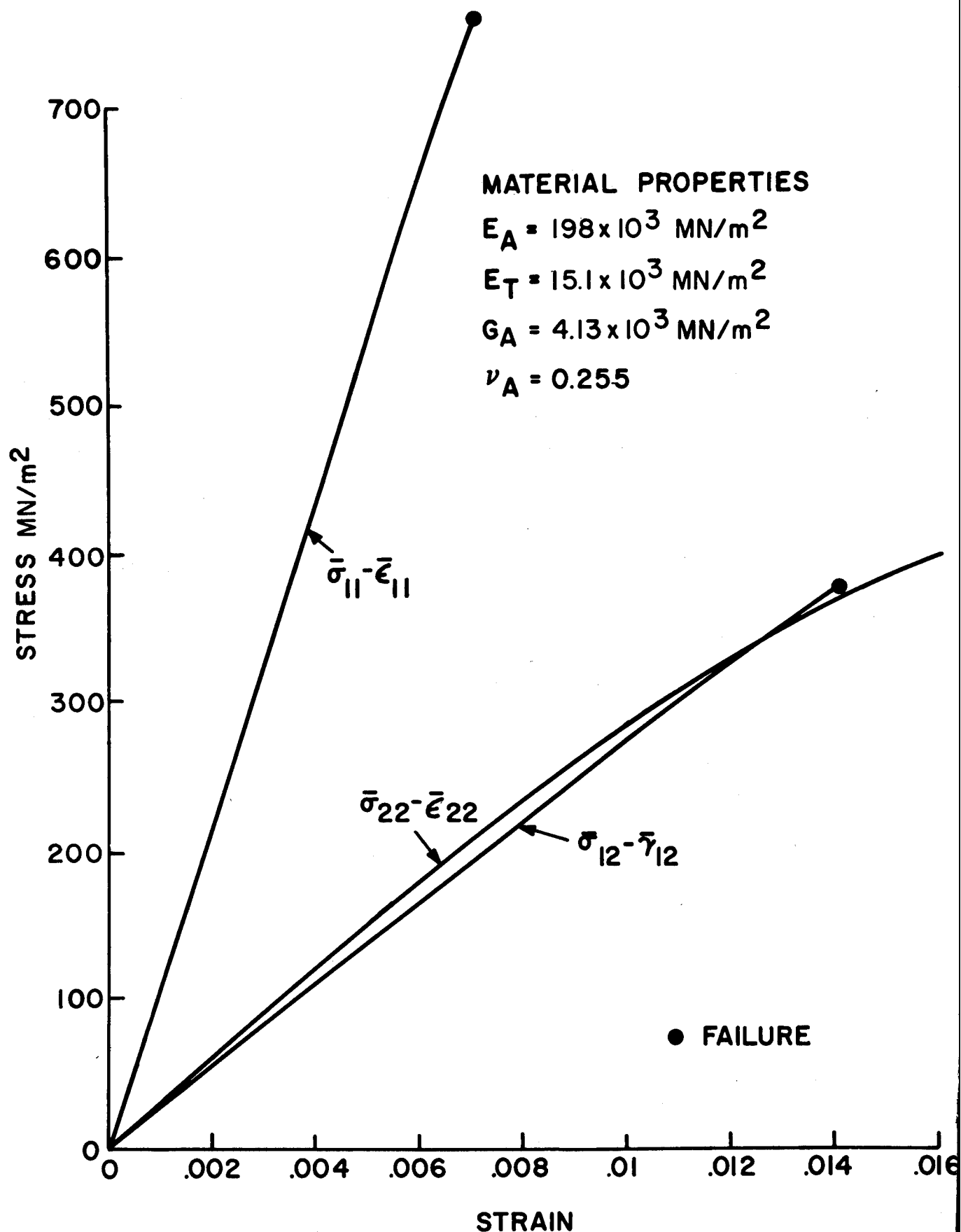


Fig. 17 - Stress-Strain curves for Axial, transverse and shear loading for  $[0^\circ/\pm 45^\circ]$  laminate.

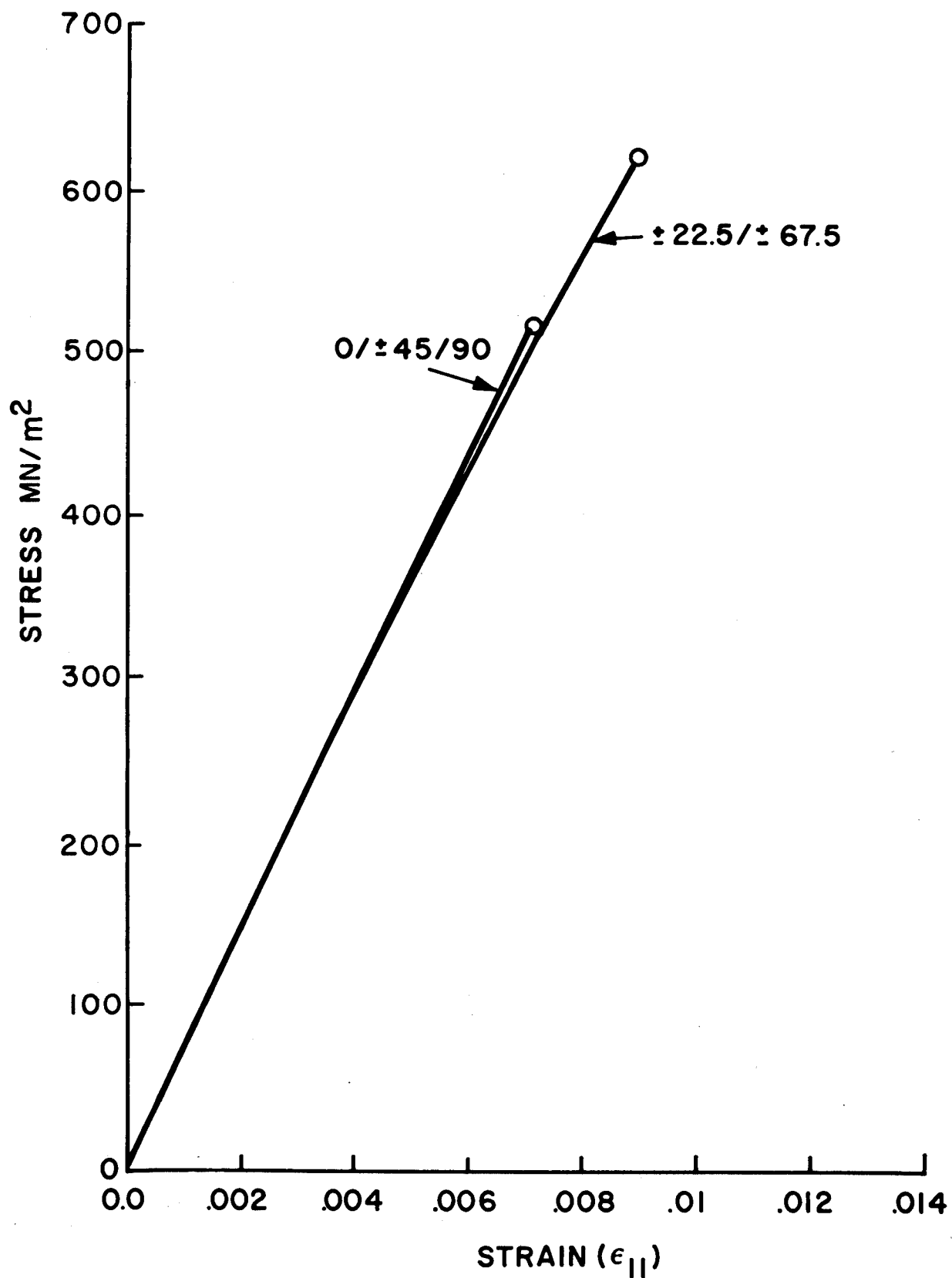


Fig. 18 - Four directional quasi-isotropic Boron/Epoxy plate under unidirectional tension in fiber direction and between fiber directions.

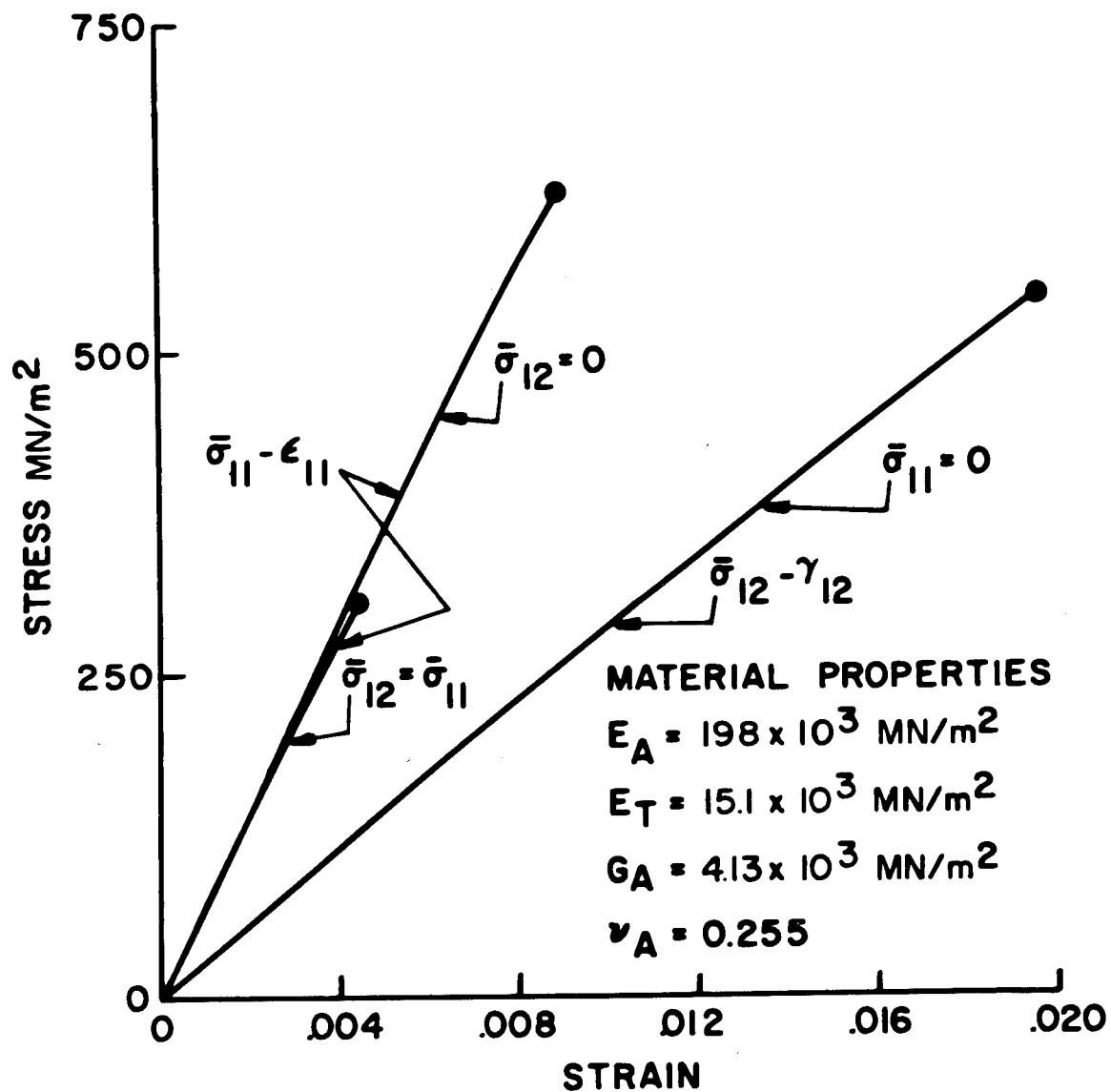


Fig. 19 - Combined stress effects on quasi-isotropic [+22.5/+67.5] laminate.

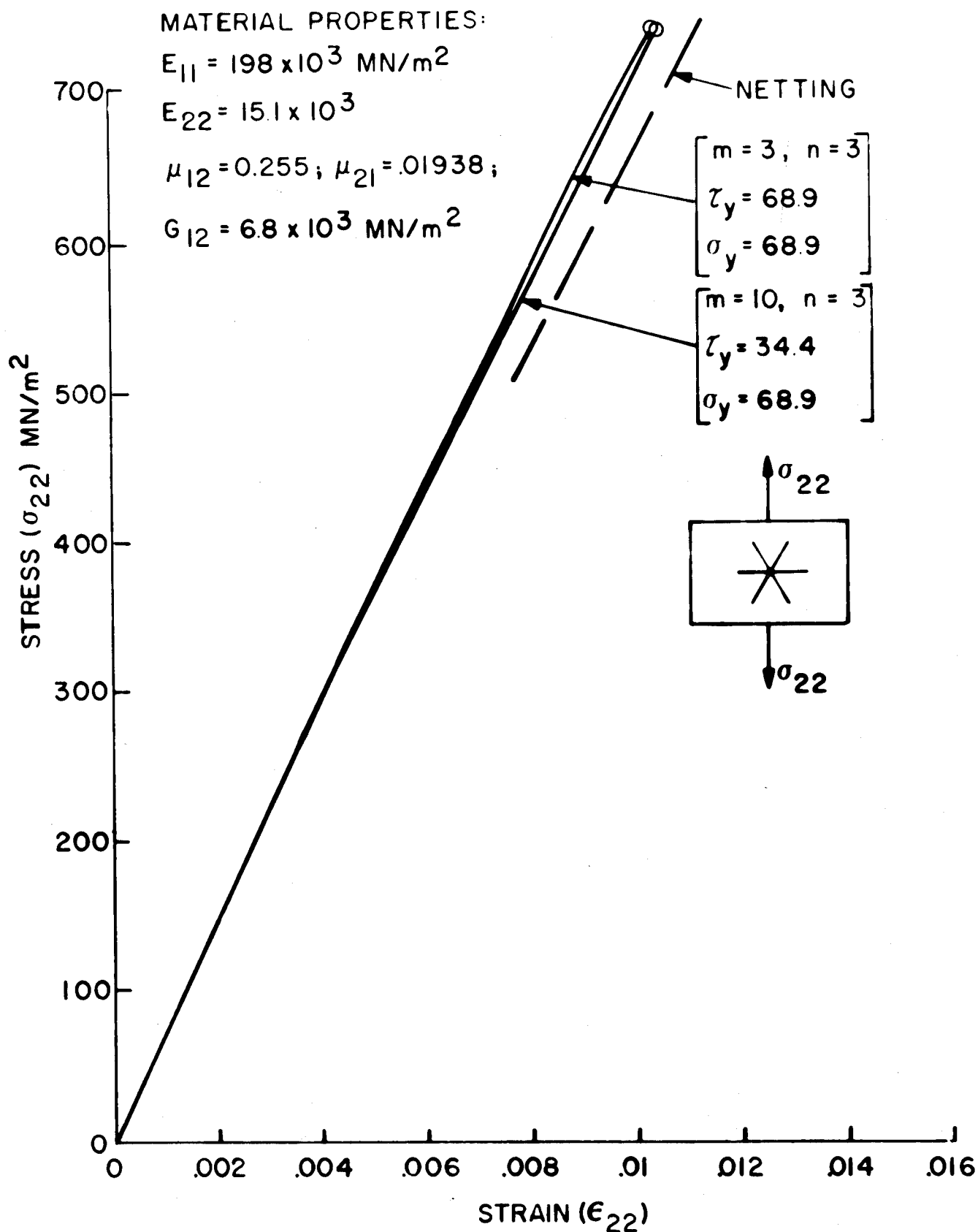


Fig. 20 - Effect of laminate inelasticity on transverse stress-strain curve for 0/+60 quasi-isotropic laminate. 93.

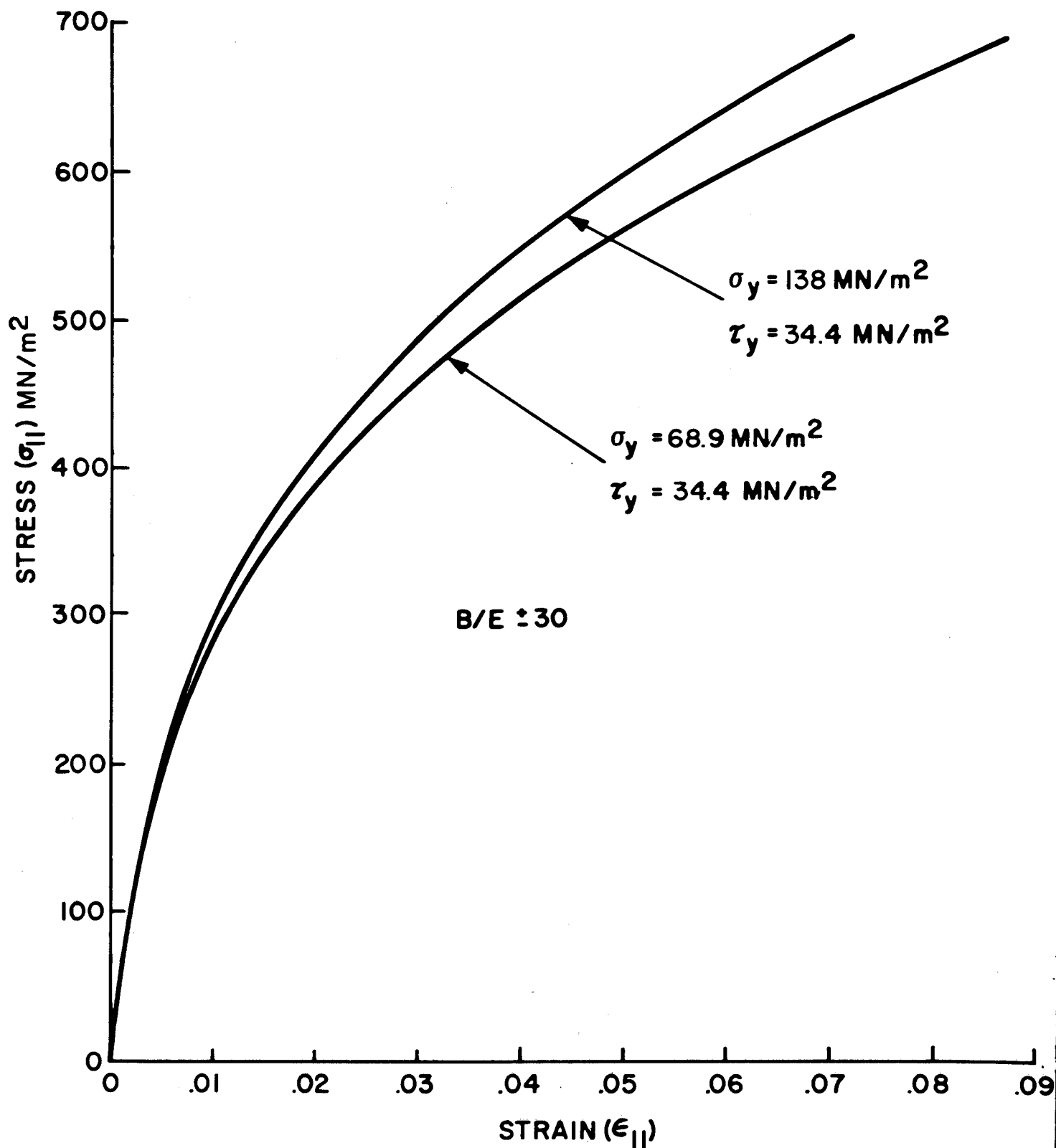


Fig. 21 - Influence of laminae inelasticity upon axial tensile stress-strain curve of  $\pm 30^\circ$  Boron/Epoxy laminate.

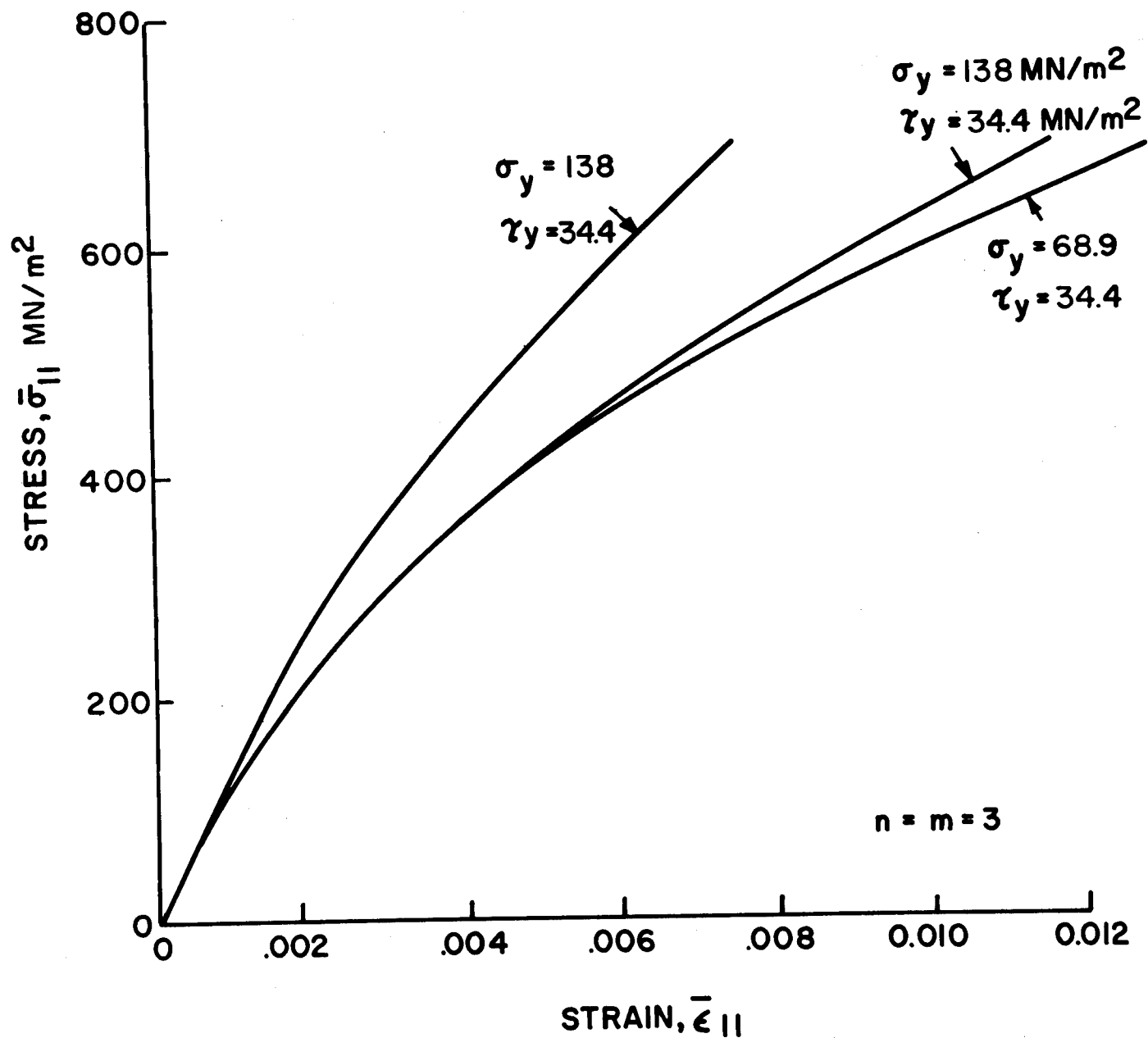


Fig. 22 - Influence of laminae inelasticity upon axial tensile stress-strain curve of  $\pm 30^\circ$  Boron/Aluminum laminate.

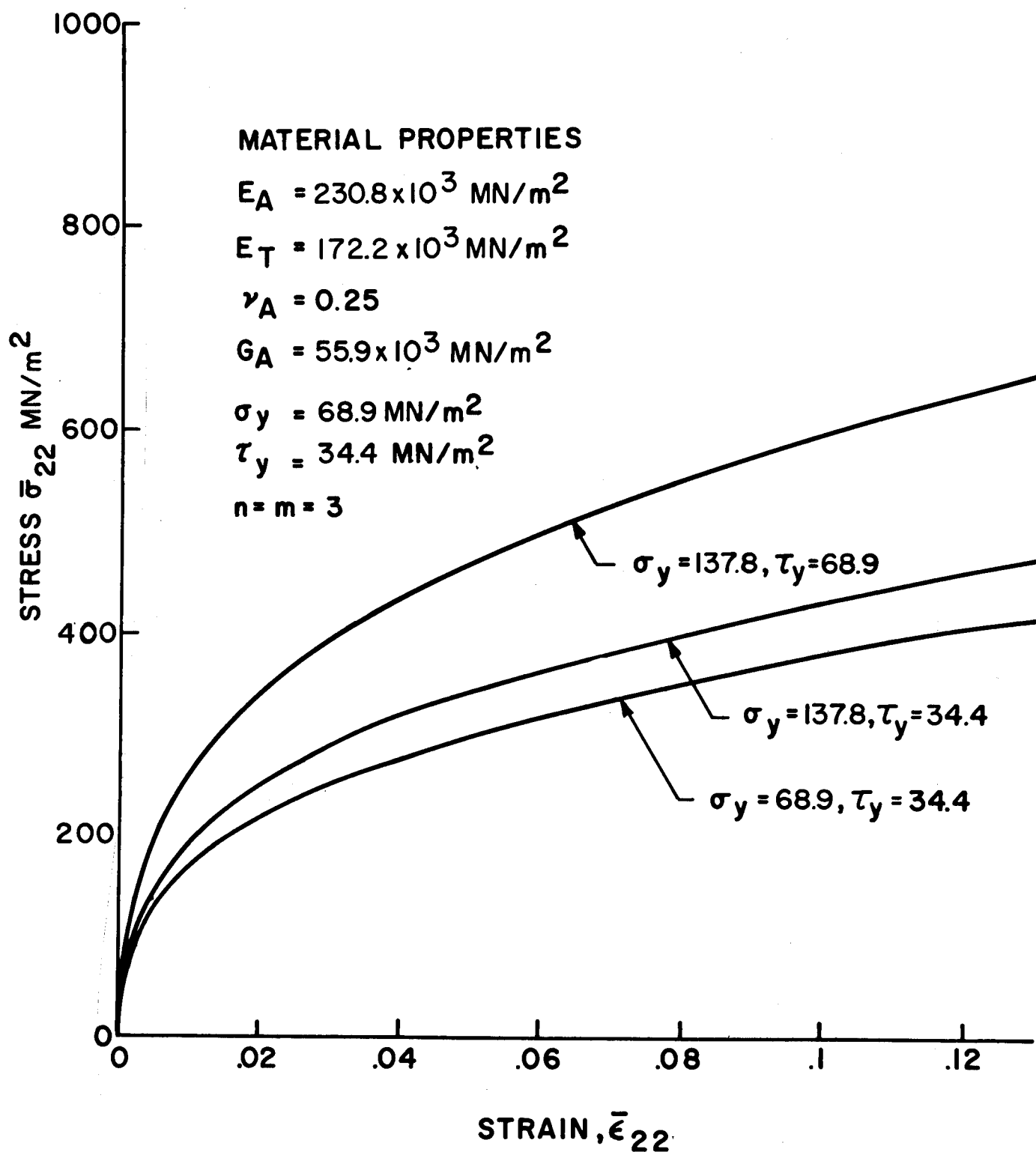


Fig. 23 - 0° Tensile stress-strain curves for  $[0_3/+45]$  B/Ep.

# MATERIAL PROPERTIES

$$E_A = 230.8 \times 10^3 \text{ MN/m}^2$$

$$E_T = 172.2 \times 10^3 \text{ MN/m}^2$$

$$\nu_A = 0.25$$

$$G_A = 55.9 \times 10^3 \text{ MN/m}^2$$

$$\sigma_y = 68.9 \text{ MN/m}^2$$

$$\tau_y = 34.4 \text{ MN/m}^2$$

$$n = m = 3$$

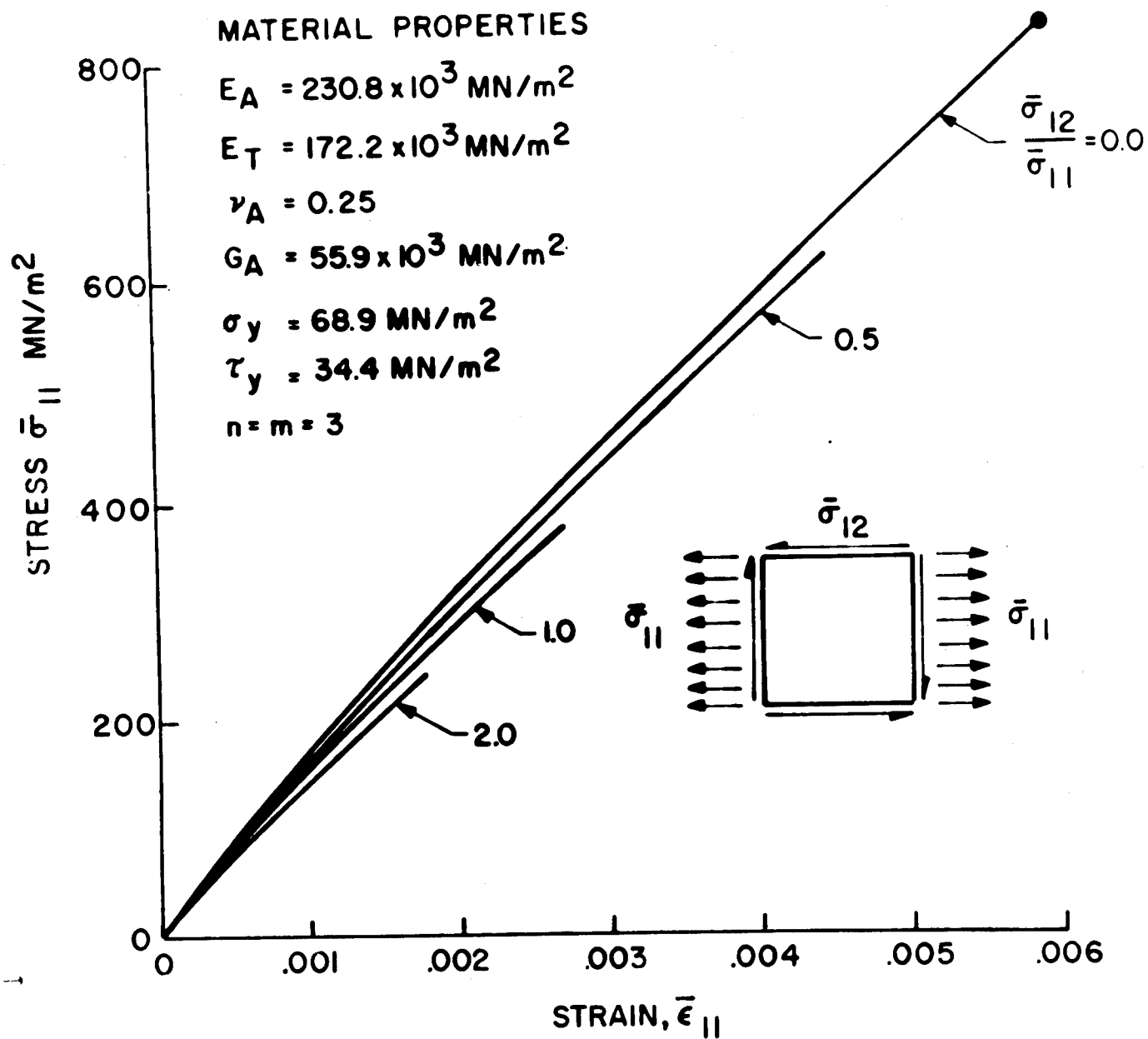


Fig. 24 - Stress-strain curves of Boron/Aluminum [0/+30] laminate under combined loading.



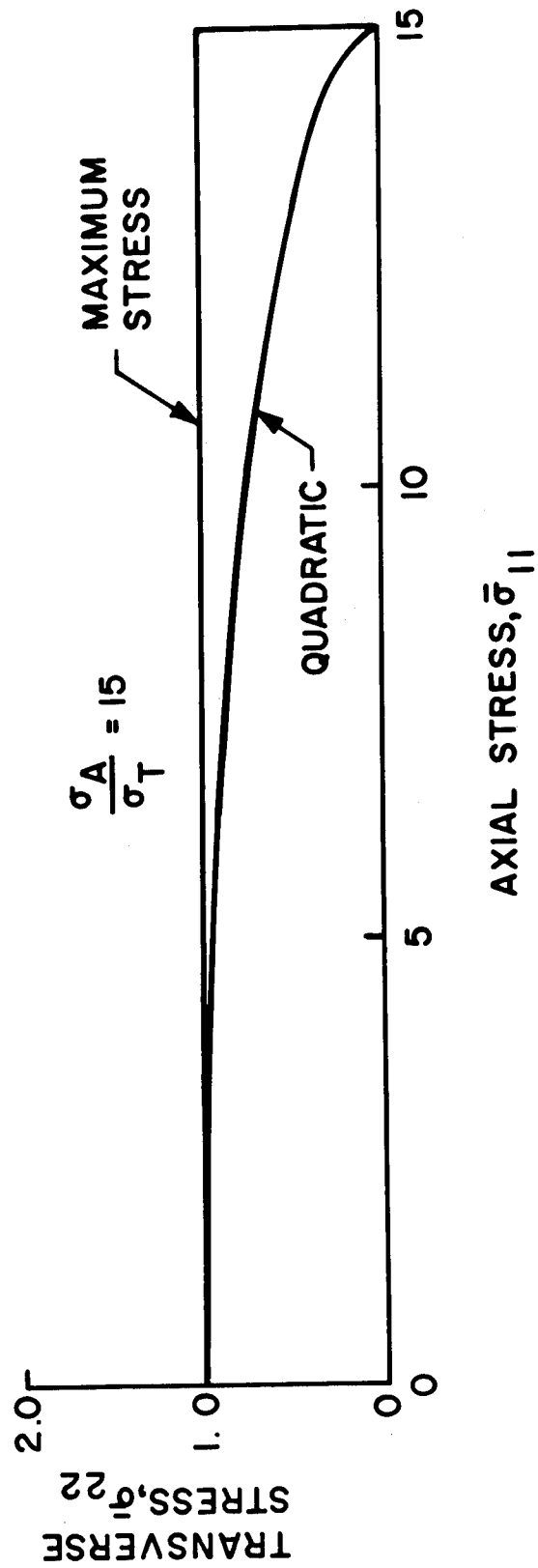


Fig. 25 - Comparison of failure criteria.

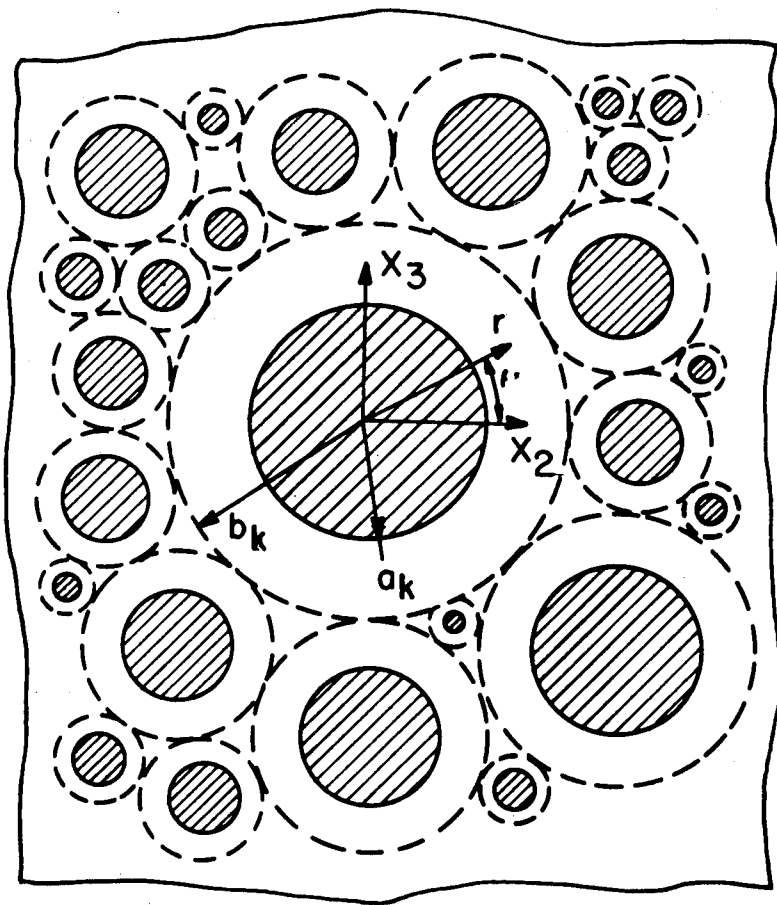


Fig. 26 - Composite Cylinder Assemblage

## C INELASTIC LAMINATE ANALYSIS

```

0001      C
0002      C
0003      C
0004      C
0005      C
0006      C
0007      C
0008      C
0009      C
0010      C
0011      C
0012      C
0013      C
0014      C
0015      C
0016      C
0017      C
0018      C
0019      C
0020      C
0021      C
0022      C
0023      C
0024      C
0025      C
0026      C
0027      C
0028      C
0029      C
0030      C
0031      C

      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION E11(20),E22(20),V12(20),V21(20),G12(20),SY(20),
     1 S11(20),S12(20),S21(20),S22(20),S44(20),T(20),
     2 A(20,20),SGC(20,1),SG(20,1),SF(20),IANG(20),
     3 SINS(20),COSS(20),SIN2(20),COS2(20)
      DIMENSION P11(20),P22(20),P12(20),PS11(20),PS22(20),
     1 PS12(20),EPS11(20),EP22(20),EPI2(20),EPS11(20),
     2 EPS22(20),EPS12(20),SGS(20),DIF(20),SG2(20),
     3 SGI(20,1),DB(20,20),BT(20),DC(20),TY(20),
     4 F(3,20),G(3,20),H(3,20)
      COMMON /ANGREF/SIN2,COS2,SINS,COSS
      COMMON /CCPARM/EP5,UPB0
      COMMON /NCPARM/II,NIT
      COMMON /TLPARM/SGR,STIFF
      COMMON /EQNTRM/EPI1,EP511,SM11,SC11
      COMMON /RAFNMT/S11,S12,S21,E22
      LOGICAL MSING,MSINGD
      INPUT
      NOTE: CONVENTION FOR V12 AND V21 ESTABLISHED BY FOLLOWING
      RELATIONSHIPS-
      S12 = -V12/E11
      S21 = -V21/E22

      WRITE(6,1500)
      READ(5,1010) NSETS
      DO 999 J = 1,NSETS
      READ(5,1010) LAY
      READ(5,1010) INP
      C
      LAMINATE OUTPUT HEADING
      WRITE(6,1509) J
      WRITE(6,1581)
      WRITE(6,1510) LAY
      C
      GO TO (20,25), INP
      C
      INPUT 1:
      20 CONTINUE
      DO 24 I = 1,LAY
      READ(5,1002) E11(I),E22(I),V12(I),V21(I)
      READ(5,1002) G12(I),SY(I),TY(I)
      READ(5,1022) T(I),IANG(I)
      24 CONTINUE
      GO TO 50
      C
      INPUT 2:
      25 CONTINUE
      WRITE(6,1561)
      WRITE(6,1563)
      WRITE(6,1565)
      DO 29 I = 1,LAY
      CALL INPUT2(E11,E22,V12,V21,G12,SY,TY,I)
      READ(5,1022) T(I),IANG(I)
      29 CONTINUE
      0031      C

```

```
0032      C      29 CONTINUE
0033      C      50 CONTINUE
0034      C      EQUATION PARAMETERS
0035      C      READ(5,1002) XN,XM
0036      C      INITIAL LCADING
0037      C      READ(5,1002) S011,S022,S012
0038      C      INCRIMENTATION PARAMETERS
0039      C      READ(5,1024) KSGM,SMLT
0040      C      LAMINATE TEST PARAMETERS
0041      C      READ(5,1002) STIFF
0042      C      READ(5,1002) SGR
0043      C      CONTROL PARAMETERS
0044      C      READ(5,1024) IT,EPS,UP8D
0045      C      READ(5,1010) INMT
0046      C      OUTPUT I.
0047      C      WRITE(6,1513)
0048      C      DO 60 I=1,LAY
0049      C      WRITE(6,1515) I,IANG(I),T(I),E11(I),E22(I),V12(I),V21(I),
0050      C      G12(I),SY(I),TV(I)
0051      C      60 CONTINUE
0052      C      WRITE(6,1516)
0053      C      WRITE(6,1517) XM
0054      C      WRITE(6,1518) XN
0055      C      WRITE(6,1520)
0056      C      WRITE(6,1583)
0057      C      WRITE(6,1595)
0058      C
0059      C      ANGLE REDUCTION ROUTINE
0060      C      CALL ANGLE(LAY,IANG)
0061      C
0062      C      INITIAL ASSIGNMENTS AND COMPUTATIONS
0063      C      TT = 0.000
0064      C      DO 100 I = 1,LAY
0065      C      TT = TT + T(I)
0066      C      100 CONTINUE
0067      C      DO 105 I=1,20
0068      C      SG(I,1) = 0.0000 00
0069      C      SGS(I) = 0.0000
0070      C      SF(I) = 1.000
0071      C      105 CONTINUE
0072      C      SM11 = SMLT*S011
0073      C      SM22 = SMLT*S022
0074      C      SM12 = SMLT*SC12
0075      C      KSG = 1
0076      C      LT1 = LAY
0077      C      N = LT1
0078      C      LP1 = LAY +1
0079      C      LT2 = LAY*2
0080      C      LT21 = LAY*2+1
0081      C      LT3 = LAY*3
0082      C      LM1 = LAY -1
0083      C
0084      C      DO 107 I = 1,LAY
0085      C      S11(I) = 1.000/E11(I)
0086      C      S12(I) = -V12(I)/E11(I)
0087      C      S21(I) = -V21(I)/E22(I)
0088      C      107 CONTINUE
0089      C
```

```

0077      110 CONTINUE
C
C      RETURN TO 110 AFTER INCREMENTING APPLIED LOAD DR
C      FROM LAMINATE TESTS 3 OR 4
C
0078      MSING = .FALSE.
0079      MSINGD = .FALSE.
0080      NIT = 0
C
0081      DO 111 K = 1,20
0082      DC(K) = 0.000
0083      SGC(K,1) = 0.000
0084      SG1(K,1) = 0.000
0085      SG2(K) = 0.000
0086      DO 111 L=1,20
0087      DB(K,L) = 0.000
0088      AL(K,L) = 0.000
0089      111 CONTINUE
0090      DO 1115 I=1,N
0091      S22(I) = 1.000/E22(I)
0092      S44(I) = 1.000/(4.000*G12(I))
0093      1115 CONTINUE
C
C      AFTER 2ND INCREMENTATION USE MULTIPLICATIVE FACTOR
C      AS INITIAL STRESS SOLUTION ESTIMATE
0094      112 CONTINUE
0095      IF(KSG.GE.3) GO TO 120
0096      DO 115 I=1,LM1
C
0097      SNS = SINS(I)
0098      CSS = CCSS(I)
0099      SN2 = SIN2(I)
0100      CS2 = CCS2(I)
C
0101      IF(IN.EQ.1) GO TO 113
0102      SN2P = SNS(I+1)
0103      CSSP = CCSS(I+1)
0104      SN2P = SIN2(I+1)
0105      CS2P = CCS2(I+1)
C
0106      113 CONTINUE
0107      A(1,I) = CSS*(I)
0108      A(1,I+N) = SAS*(I)
0109      A(1,I+2*N) = -SN2*(I)
C
0110      A(2,I) = SAS*(I)
0111      A(2,I+N) = CSS*(I)
0112      A(2,I+2*N) = SN2*(I)
C
0113      A(3,I) = SN2*(I)/2.000
0114      A(3,I+N) = -SN2*(I)/2.000
0115      A(3,I+2*N) = CS2*(I)
0116      IF(IN.EQ.1) GO TO 116
C
0117      A(3*I+1,I) = -S11(I)*CSS -S21(I)*SNS
0118      A(3*I+1,I+1) = S11(I)*CSSP+S21(I+1)*SNSP
0119      A(3*I+1,I+N) = -S12(I)*CSS -S22(I)*SNS
0120      A(3*I+1,I+N+1) = S12(I)*CSSP+S22(I+1)*SNSP
0121      A(3*I+1,I+2*N) = 2.000*S44(I)*SN2
0122      A(3*I+1,I+2*N+1) = -2.000*S44(I+1)*SN2P

```

```

0123 A(3*I+2,1) J = -S11(I) J*SN5 -S21(I) J*CS5
0124 A(3*I+2,1+1) J = S11(I+1)*SN5P+S21(I+1)*CS5P
0125 A(3*I+2,1+N) J = -S12(I) J*SN5 -S22(I) J*CS5
0126 A(3*I+2,1+N+1) J = S12(I+1)*SN5P+S22(I+1)*CS5P
0127 A(3*I+2,1+2*N) J = -2.0D0*S44(I) J*SN2
0128 A(3*I+2,1+2*N+1) J = 2.0D0*S44(I+1)*SN2P
C
0129 A(3*I+3,1) J = -(S11(I) J-S21(I) J)*SN2 /2.0D0
0130 A(3*I+3,1+1) J = -(S11(I+1)-S21(I+1))*SN2P/2.0D0
0131 A(3*I+3,1+N) J = -(S12(I) J-S22(I) J)*SN2 /2.0D0
0132 A(3*I+3,1+N+1) J = -(S12(I+1)-S22(I+1))*SN2P/2.0D0
0133 A(3*I+3,1+2*N) J = -2.0D0*S44(I) J*CS2
0134 A(3*I+3,1+2*N+1) J = 2.0D0*S44(I+1)*CS2P
C
0135 C 115 CONTINUE
C
0136 A(1,N) J = CS5P*(I,N)
0137 A(1,2*N) J = SN5P*(I,N)
0138 A(1,3*N) J = -SN2P*(I,N)
0139 A(2,N) J = SN5P*(I,N)
0140 A(2,2*N) J = CS5P*(I,N)
0141 A(2,3*N) J = SN2P*(I,N)
0142 A(3,N) J = SN2P*(I,N)/2.0D0
0143 A(3,2*N) J = -SN2P*(I,N)/2.0D0
0144 A(3,3*N) J = CS2P*(I,N)
0145 IF(MSING) GO TO 117
C
0146 C 116 CONTINUE
C
0147 C INVERT MATRIX A
C CALL INVERTC(A,20,LT3,DET,1.0D-15,IRANK)
C
0148 C CHECK FOR SINGULAR MATRIX
C IF(IRANK.EQ.LT3) GO TO 118
0149 MSING = .TRUE.
0150 WRITE(6,1420) IRANK,DET
0151 GO TO 112
0152 C 117 CONTINUE
0153 WRITE(6,1425) ((A(K1,L1),L1=1,LT3),K1=1,LT3)
0154 MSING = .FALSE.
0155 GO TO 999
0156 C 118 CONTINUE
C
0157 C B.C. VECTOR
C SGO(1,1) = S011
0158 SGO(2,1) = S022
0159 SGO(3,1) = S012
0160 CALL MXMULD(A,SGO,SG,20,20,1,LT3,LT3,1)
C
0161 C RESET STRESS = 0, IF RELATIVE STRESS < 1.0D-06
C CALL RESET(LT3,SG,1.0D-06)
0162 GO TO 126
C
0163 C (MULTIPLICATIVE FACTOR) X (SOLUTION FROM PREVIOUS INCREMENT)
C 120 CONTINUE
0164 DO 122 I=1,LT3
0165 SG(I,1) = SF(I)*SGS(I)
0166 122 CONTINUE
0167 GO TO 126
C

```

```

C RETURN TO 125 FOR NEXT ITERATION STEP
0168 125 CONTINUE
0169 NIT = NIT + 1
0170 126 CONTINUE
0171 DO 127 I=1,LT3
0172 SG(I,1) = SG(I,1)
0173 SI(I) = 0.0D 00
0174 127 CONTINUE
C
C 130 CONTINUE
0175 DO 1305 K=1,LT3
0176 DO 1305 L=1,LT3
0177 DB(K,L) = 0.0D 00
0178 1305 CONTINUE
0179 IF(N.EC.1) LM1=1
0180 DO 151 I=1,LM1
0181 CALL NRTRM (LAY,SG,F,G,H,G12,SY,TY,XN,XM,I)
0182 CALL NRTRM (LAY,SG,F,G,H,G12,SY,TY,XN,XM,I+1)
0183
C
0184 SNS = SINS(I)
0185 CSS = COS(I)
0186 SN2 = SIN2(I)
0187 CS2 = COS2(I)
C
0188 IF(N.EC.1) GO TO 131
0189 SN2P = SINS(I+1)
0190 CSSP = COS(I+1)
0191 SN2P = SIN2(I+1)
0192 CS2P = COS2(I+1)
C
0193 131 CONTINUE
0194 DB(I,I) = CSS*(I)
0195 DB(I,I+N) = SNS*(I)
0196 DB(I,I+2*N) = -SN2*(I)
0197 CB(I,I) = SNS*(I)
0198 DB(I,I+N) = CSS*(I)
0199 DB(I,I+2*N) = SN2*(I)
0200 DB(I,I) = SN2*(I)/2.
0201 DB(I,I+N) = -SN2*(I)/2.
0202 DB(I,I+2*N) = CS2*(I)
0203 IF(N.EC.1) GO TO 161
C
0204 CB(I,I+1,I) = -F(I,I)
0205 DB(I,I+1,I+1) = F(I,I+1)
0206 DB(I,I+1,I+N) = -F(I,I)
0207 DB(I,I+1,I+N+1) = F(I,I+1)
0208 DB(I,I+1,I+2*N) = -F(I,I)
0209 CB(I,I+1,I+2*N+1) = F(I,I+1)
C
0210 DB(I,I+2,I) = -G(I,I)
0211 DB(I,I+2,I+1) = G(I,I+1)
0212 DB(I,I+2,I+N) = -G(I,I)
0213 DB(I,I+2,I+N+1) = G(I,I+1)
0214 DB(I,I+2,I+2*N) = -G(I,I)
0215 DB(I,I+2,I+2*N+1) = G(I,I+1)
C
0216 CB(I,I+3,I) = -H(I,I)
0217 DB(I,I+3,I+1) = H(I,I+1)
0218 DB(I,I+3,I+N) = -H(I,I)
0219 CB(I,I+3,I+N+1) = H(I,I+1)

```

```
0220 DB(3*I+3,I+2*N) = -H(3,I)
0221 DB(3*I+3,I+2*N+1) = H(3,I+1)

0222 151 CONTINUE
0223 DB(1,N) = CSSP*I(N)
0224 DB(1,2*N) = SNSP*I(N)
0225 DB(1,3*N) = -SN2P*I(N)
0226 DB(2,N) = SNSP*I(N)
0227 DB(2,2*N) = CSSP*I(N)
0228 DB(2,3*N) = SN2P*I(N)
0229 DB(3,N) = SN2P*I(N)/2.
0230 DB(3,2*N) = -SN2P*I(N)/2.
0231 DB(3,3*N) = CS2P*I(N)
0232 IF(MSINGD) GO TO 167
0233 161 CONTINUE

C
C UNDERFLOW CHECK ON MATRIX
0234 DO 163 K=1,LT3
0235 DO 163 L=1,LT3
0236 IF(DABS(CB(K,L))-LT.1.0D-40) DB(K,L) = 0.0D 00
0237 163 CONTINUE

C
C INVERT MATRIX DB
0238 CALL INVRTD(CB,20,LT3,DET,1.0D-12,IFRANK)

C
C CHECK FOR SINGULAR MATRIX
0239 IF(IRANK.EQ.LT3) GO TO 168
0240 MSINGD = .TRUE.
0241 WRITE(6,1420) IFRANK,DET
0242 GO TO 130
0243 167 CONTINUE
0244 WRITE(6,1425) ((DB(K,L),L=1,LT3),K=1,LT3)
0245 MSINGD = .FALSE.
0246 GO TO 599
0247 168 CONTINUE

C
C
C
0248 DC(1) = -S011*TI
0249 DC(2) = -S022*TI
0250 DC(3) = -S012*TI

C
DO 6 I=1,LAY
0251 SNS = SINS(I)
0252 CSS = COS(I)
0253 SN2 = SIN2(I)
0254 CS2 = COS2(I)
0255 DC(1) = DC(1) + SG(1,I)*CSS*I(I) + SG(LAY+1,I)*SNS*I(I)
0256 -SG(2*LAY+1,I)*SN2*I(I)
0257 DC(2) = DC(2) + SG(1,I)*SNS*I(I) + SG(LAY+1,I)*CSS*I(I)
0258 +SG(2*LAY+1,I)*SN2*I(I)
0258 DC(3) = DC(3) + SG(1,I)*TI(I)*SN2/2. - SG(LAY+1,I)*TI(I)*
0259 SN2/2. + SG(2*LAY+1,I)*CS2*TI(I)
6 CONTINUE

C
C REDEFINE COEFFICIENTS S22 AND S44
DO 7 K=1,N
0260 T125 = (SG(K+2*N,1)/TY(K))**2
0261 TS225 = (SG(K+N,1)/SY(K))**2
0262 S125 = T125 + TS225
0263 S22(K) = (1.0D0+S125**((XN-1.)/2.))/E22(K)
0264
```



```

0265 S44(K) = (1.000*S125*((IXM-1.)/2.))/(4.000*G12(K))
0266 7 CONTINUE
C
0267 DO 8 K=1,LM1
0268 SNS = SINS(K)
0269 CSS = CCSS(K)
0270 SN2 = SIN2(K)
0271 CS2 = CCS2(K)
0272 SN2P = SINS(K+1)
0273 CSSP = CCSS(K+1)
0274 SN2P = SIN2(K+1)
0275 CS2P = COS2(K+1)
C
0276 DC(3+3*K-2) = -(S11(K)*CSS+S21(K)*SNS)*SG(K,1) -
1 SG(LAY+K,1)*(S12(K)*CSS+S22(K)*SNS) + 2.*S44(K)*SN2*
2 SG(2*LAY+K,1) + (S11(K+1)*CSSP+S21(K+1)*SN2P)
3 *SG(K+1,1) + (S12(K+1)*CSSP+S22(K+1)*SN2P)*
4 SG(LAY+K+1,1) - 2.*S44(K+1)*SG(2*LAY+K+1,1)*
5 SN2P
0277 DC(3+3*K-1) = -(S11(K)*SNS+S21(K)*CSS)*SG(K,1) - (S12(K)*SNS+
1 S22(K)*CSS)*SG(LAY+K,1) - 2.*S44(K)*SN2*
2 SG(2*LAY+K,1) +
3 (S11(K+1)*SN2P+S21(K+1)*CSSP)*SG(LAY+K+1,1) + (S12(K+1)*
4 SNSP+S22(K+1)*CSSP)*SG(LAY+K+1,1) + 2.*S44(K+1)*SN2P
5 *SG(2*LAY+K+1,1)
0278 DC(3+3*K) = -(S11(K)-S21(K))*SN2*SG(K,1)/2. - (S12(K)-S22(K))*
1 SN2*SG(LAY+K,1)/2. - 2.*S44(K)*CS2*SG(2*LAY+K,1) +
2 (S11(K+1)-S21(K+1))*SN2P*SG(K+1,1)/2. + (S12(K+1)-
3 S22(K+1))*SN2P*SG(LAY+K+1,1)/2. + 2.*S44(K+1)*CS2P*
4 SG(2*LAY+K+1,1)
0279 8 CONTINUE
C
0280 DO 9 I=1,LT3
0281 DC(I) = -DC(I)
0282 9 CONTINUE
C
0283 DO 1 I=1,LT3
0284 DO 1 K=1,LT3
0285 1 BT(I) = BT(I) + DB(I,K)*DC(K)
C
0286 DO 15 I=1,LT3
0287 SG(I,1) = SG(I,1) + BT(I)
0288 15 CONTINUE
C
0289 RESET STRESS = 0. IF RELATIVE STRESS < 1.0E-06
0290 CALL RESET(LT3,SG,1.0D-06)
IF(NIT.EQ.0) GO TO 125
C
0291 CONVERGENCE CHECK
0292 CALL CONVR(LAY,SG,SG1,KSG,&125,&900)
500 CONTINUE
C
0293 STRAIN COMPUTATIONS
DO 540 I = 1,LAY
0294 SNS = SINS(I)
0295 CSS = CCSS(I)
0296 SN2 = SIN2(I)
0297 CS2 = COS2(I)
C
0298 P11(I) = S11(I)*SG(I,1) + S12(I)*SG(I+N,1)
0299 TL25 = (SG(I+2*N,1)/TV(I))*#2

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0300      IS22S = (SG(I+N,1)/SY(I))*2
0301      S12S = T12S + IS22S
0302      P22(I) = S21(I)*SG(I,1) + SG(I+N,1)/E22(I)*(1.0D0+S12S**((IXN
1          -1.)/2.))
0303      PL2(I) = SG(I+2*N,1)/I2*(I2*GL2(I))*((1.0D0+S12S**((IXN-1.)/2.))
0304      520 CONTINUE
C
0305      EP11(I) = P11(I)*CSS + P22(I)*SNS - P12(I)*SN2
0306      EP22(I) = P11(I)*SNS + P22(I)*CSS + P12(I)*SN2
0307      EP12(I) = (P11(I)-P22(I))*SN2/2.0D0 + P12(I)*CS2
C
0308      540 CONTINUE
C
0309      C LAMINATE TESTS
CALL LAMTSTLAY,SG,S6S,KSG,KSGM)
C
C OUTPUT II.
C
0310      730 CONTINUE
0311      WRITE(6,1525)
0312      WRITE(6,1527) S011
0313      WRITE(6,1528) S022
0314      WRITE(6,1529) S012
0315      WRITE(6,1735) NIT
0316      WRITE(6,1536)
0317      WRITE(6,1537)
0318      WRITE(6,1538)
0319      DO 750 I = 1,LAY
0320      WRITE(6,1550) I,SG(I,1),SG(I+N,1),SG(I+2*N,1),
1          EP11(I),EP22(I),EP12(I),P11(I),P22(I),P12(I)
C
0321      750 CONTINUE
0322      WRITE(6,1588)
C
0323      C IF(KSGM-KSG) 790,790,760
C
C MULTIPLICATIVE FACTOR FOR INITIAL ESTIMATE OF SUCCEEDING
C INCREMENTATION
C
0324      760 CONTINUE
0325      GO TO (762,766), INNT
C
C RATIO OF PREVIOUS SOLUTIONS
C
0326      762 CONTINUE
0327      DO 765 I=1,LT3
0328      IF(KSG.EQ.1) GO TO 770
0329      IF(S6S(I).EQ.0.0D0) GO TO 763
0330      SF(I) = SG(I,1)/S6S(I)
0331      GO TO 765
C
0332      763 CONTINUE
0333      SF(I) = 1.0D0
0334      765 CONTINUE
0335      GO TO 770
C
0336      766 CONTINUE
0337      DO 768 I=1,LT3
0338      IF(KSG.LT.2) GO TO 767
0339      IF(SG(I,1).EQ.0.0D0) GO TO 767
0340      KSG = KSG + 1
0341      CONS = VKSG*(VKSG-27/(VKSG-1))*2
0342      SF(I) = 1.0D0 + CONS*(SG(I,1)-S6S(I))/SG(I,1)
0343      GO TO 768
0344      767 CONTINUE

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0345      SF(1) = 1.000
0346      768 CONTINUE
C
0347      C      STORE STRESS AND STRAIN VALUES
0348      770 CONTINUE
0349      DO 775 I = 1,LAY
0350      SCS(I + N) = SG(I + N,1)
0351      SCS(I + 2*N) = SG(I + 2*N,1)
0352      PS11(I) = P11(I)
0353      PS22(I) = P22(I)
0354      PS12(I) = P12(I)
0355      EPS11(I) = EP11(I)
0356      EPS22(I) = EP22(I)
0357      EPS12(I) = EP12(I)
0358      775 CONTINUE
C
0359      C      INCREMENT APPLIED LOADING
0360      KSG = KSG + 1
0361      S011 = SC11 + SM11
0362      S022 = SC22 + SM22
0363      S012 = SC12 + SM12
0364      GO TO 110
C      790 CONTINUE
0365      C      900 CONTINUE
0366      C      999 CONTINUE
0367      WRITE(6,1999)
C
C
C
0368      1002 FORMAT (5D15.5)
0369      1010 FORMAT (5I5)
0370      1022 FORMAT (D15.5,15)
0371      1024 FORMAT (15,5D15.5)
C
0372      1420 FORMAT (// 'MATRIX IS SINGULAR OF RANK = ',I2,
0373      1425 FORMAT (6D15.5)
C
0374      1500 FORMAT (' INELASTIC LAMINATE ANALYSIS'////)
0375      1505 FORMAT (1H1,'LAMINATE ',I2)
0376      1510 FORMAT (//// 'NUMBER OF LAYERS = ',I2)
0377      1513 FORMAT ('// LAYER THETA',7X,'T',15X,'E11',11X,'E22',11X,
0378      1515 FORMAT (14,5X,13,2X,D12.5,4X,7(2X,D12.5))
0379      1515 FORMAT (// 'EQUATION PARAMETERS')
0380      1517 FORMAT ('// EXPONENT M = ',D12.5)
0381      1518 FORMAT ('// EXPONENT N = ',D12.5)
0382      1520 FORMAT (// 'LOADING INCREMENTATION')
0383      1525 FORMAT ('// EXTERNAL APPLIED LOADING')
0384      1527 FORMAT ('// F X = ',D13.5)
0385      1528 FORMAT ('// F Y = ',D13.5)
0386      1529 FORMAT ('// F XY = ',D13.5)
0387      1536 FORMAT (122X,'STRESS',38X,'STRAIN',37X,'STRAIN')
0388      1537 FORMAT (162X,'(LAMINATE AXES)',429X,'(LAYER AXES)')
0389      1538 FORMAT ('// LAYER',4X,'SC1 X',8X,'SGM Y',8X,'SGM XY',11X,'EPS X',
0390      1540      8X,'EPS Y',8X,'EPS XY',11X,'EPS X',8X,'EPS Y',8X,
0391      1545      'EPS XY')
0392      1550 FORMAT (14,1X,2(3D13.5,4X),3D13.5)

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0391 1561 FORMAT (//57X,'MATERIAL PROPERTIES')  
 0392 1563 FORMAT (/34X,'F19E9',54X,'MATH:IX')  
 0393 1565 FORMAT (/11X,'F1',11X,'MU',12X,'C',12X,'V',12X)/)

C 1735 FORMAT (/1 SOLUTION FOR STRESS CONVERGES WITHIN 14,  
 1 ITERATIONS)/)

C 1581 FORMAT ('-----')  
 0396 1583 FORMAT ('-----')  
 0397 1586 FORMAT (/62X,'-----')  
 0398 1591 FORMAT (1E0)  
 0399 1599 FORMAT (1F1)  
 0400 STOP  
 0401 END

0001

SUBROUTINE ANGLE(LAY, IANG)

C

C

C REDUCE ANGLES TO VALUES BETWEEN 0 AND AND PI/4 FOR

C COMPUTING SIN AND COS

C

C

0002 IMPLICIT REAL\*8 (A-H,O-Z)

0003 DIMENSION SINS(20),COSS(20),SIN2(20),COS2(20),IANG(20)

0004 COMMON /ANGRED/SIN2,COS2,SINS,COSS

0005 DO 72 I = 1,LAY

0006 IANG2 = 2\*IANG(I)

0007 ANG = IANG(I)

0008 ANG2 = IANG2

0009 RAD = ANG/57.29577951300

0010 RAD2 = ANG2/57.29577951300

0011 IANG = IABS(IANG(I))

0012 IF(IANG.EQ.0) GO TO 66

0013 IF(IANG.LE.90) GO TO 62

0014 SINS(I) = DCOS(0.000)\*\*2

0015 COSS(I) = DSIN(0.000)\*\*2

0016 SIN2(I) = DSIN(0.000)

0017 COS2(I) = -DCOS(0.000)

0018 GO TO 72

0019 62 CONTINUE

0020 ISGN = IANG(I)/IANG

0021 SGN = ISGN

0022 IF(IANG.LE.45) GO TO 64

0023 SIN2(I) = DCOS(0.000)\*SGN

0024 COS2(I) = DSIN(0.000)

0025 GO TO 68

0026 64 CONTINUE

0027 IF(IANG.LT.45) GO TO 66

0028 IRDA = 2\*IANG-90

0029 RDA = IRDA

0030 RDA = RDA/57.29577951300

0031 SIN2(I) = SGN\*DCOS(SGN\*RDA)

0032 COS2(I) = -SGN\*DSIN(SGN\*RDA)

0033 GO TO 68

0034 66 CONTINUE

0035 SIN2(I) = DSIN(RAD2)

0036 COS2(I) = DCOS(RAD2)

0037 68 CONTINUE

0038 SINS(I) = DSIN(RAD)\*\*2

0039 COSS(I) = DCOS(RAD)\*\*2

0040 72 CONTINUE

0041 RETURN

0042 END

```

0001      SUBROUTINE CONVP(LAY,SG,SG1,KS,G,W)
      C
      C
0002      IMPLICIT REAL*8 (A-F,G-Z)
0003      DIMENSION SG(20,1),SG1(20,1),DIF(20)
0004      COMMON /CCFARM/EPS,UPBD
      C
      C
0005      ICON = 1
0006      N = LAY
0007      LT3 = LAY*3
      C
      C
0008      CONVERGENCE CHECK
0009      DO 375 J3=1,LT3
0010      SUB = CABS(SG(J3,1))-DABS(SG1(J3,1))
0011      IF (SG1(J3,1)-EQ,0.000) GO TO 330
0012      DIF(J3) = DABS(SUB/SG1(J3,1))
0013      GO TO 335
0014      330 CONTINUE
0015      DIF(J3) = SUB
0016      335 CONTINUE
0017      IF (DIF(J3).GT.EPS) GO TO 340
      GO TO 375
      C
      C
0018      ITERATION CHECK
0019      340 CONTINUE
0020      IF ((NIT-IT).NE.0) GO TO 350
0021      ICON = 3
0022      IF (DIF(J3).LE.UPBD) GO TO 375
0023      ICON = 4
      GO TO 375
      C
      C
0024      DIVERGENCE CHECK
0025      350 IF (DIF(J3).LE.UPBD) GO TO 370
0026      ICON = 4
      GO TO 375
      C
      C
0027      370 CONTINUE
0028      ICON = 2
0029      375 CONTINUE
      C
0030      GO TO (500,400,382,386),ICON
      C
      C
0031      NCA-CONVERGENCE DUMP
0032      382 CONTINUE
0033      WRITE(6,1720)
0034      WRITE(6,1722) EPS
0035      GO TO 395
0036      386 CONTINUE
0037      WRITE(6,1730)
0038      WRITE(6,1722) UPBD
0039      395 CONTINUE
0040      NITP = NIT - 1
0041      WRITE(6,1741) NIT,NITP
0042      WRITE(6,1742)
0043      DO 397 I=1,LAY
0044      397 CONTINUE
0045      RETURN 2

```



SUBROUTINE LAMTST(LAY,SG,SCS,KSG,KSGM)

0001

C

```
0002      IMPLICIT      REAL*8 (A-H,O-Z)
0003      DIMENSION      EPLI(20),EPSLI(20),SG(20,1),SCS(20)
0004      COMMON          /TLPARM/SGR,STIFF
                        /EUNJRN/EPLI,EPSLI,SPLI,SCLI
```

C

TEST 1: STIFFNESS TEST

550 CONTINUE

DO 675 I = 1,LAY

IF(KSG,FQ,1) GO TO 560

IF(SOLI,EG,0.050) GO TO 560

RATIO = DABS(SMII/(EPLI(1)-EPSLI(1)))

IF(RATIO,LT,STIFF) GO TO 677

C

TEST 2: TEST FOR BOUND CN SGM X

560 CONTINUE

IF(CABS(SGLI,1),GT,SCS) GO TO 679

C

C

675 CONTINUE

C

C

WRITE(6,1995)

RETURN

677 LFAIL = 1

KSG = KSGM

GO TO 700

675 LFAIL = 2

KSG = KSGM

GO TO 700

C

C

700 CONTINUE

GO TO (701,703), LFAIL

701 WRITE(6,1450)

GO TO 725

703 WRITE(6,1452)

GO TO 725

725 WRITE(6,1495)

C

1450 FORMAT (/// LAMINATE HAS FAILED; STIFFNESS TEST FAILURE\*)

1452 FORMAT (/// LAMINATE HAS FAILED; SGM X EXCEEDS MAXIMUM\*)

1495 FORMAT (/// AT FAILURE\*)

1995 FORMAT (1P0)

C

RETURN

END

0034



```

0001 SUBROUTINE NRTRM(LAY,SC,EAG,EAGL2,SY,TV,XN,XM,1)
      C
      C
      IMPLICIT REAL*8(A-H,C-Z)
      DOUBLE PRECISION DABS
      DIMENSION S11(20),S12(20),S21(20),E22(20),G12(20),
     1 S1S(20),COS2(20),S1N2(20),COS2(20),SG12(20),SG12(20),
     2 F(3,20),G(3,20),H(3,20),TV(20),SY(20)
      COMMON /RAENNY/S11,S12,S21,E22
      C
      1 /ANGPED/SIN2,COS2,S1N2,COS2
      C
      N = LAY
      T12S = (SG11+2*N,1)/TV(11)**2
      T12D = SG11+2*N,1)/TV(11)**2
      T22S = (SG11+N,1)/TV(11)**2
      T22D = SG11+N,1)/TV(11)**2
      VAL = SY(11)*1.00D-60
      RAT = SG11+2*N,1)*SG11+N,1)/SY(11)
      VAL = CABS(VAL)
      RAT = CABS(RAT)
      IF(RAT.LE.VAL) RAT = 0.0D 00
      C12S = RAT/SY(11)
      C12T = SG11+2*N,1)*SG11+N,1)/TV(11)**2
      TS22S = (SG11+N,1)/SY(11)**2
      S12S = T12S + TS22S
      C
      SNS = S1N2(1)
      CSS = COS2(1)
      SN2 = S1N2(1)
      CS2 = COS2(1)
      C
      F(1,1) = S11(1)*CSS + S21(1)*SNS
      F(2,1) = S12(1)*CSS + SNS/E22(1) + SNS/E22(1)*S12S**((XN-1,1)/2.)
     1 + SNS*TS22S*(XN-1,1)/E22(1)*S12S**((XN-3,1)/2.)
     2 - SN2*(XN-1,1)/2.*G12(1)*S12S**((XN-3,1)/2.)*C12S
      F(3,1) = -SN2/(2.*G12(1)) - SN2/(2.*G12(1))*S12S**((XN-1,1)/2.)
     1 - SN2*TS22S*(XN-1,1)/(2.*G12(1))*S12S**((XN-3,1)/2.)
     2 + C12T*(XN-1,1)/E22(1)*S12S**((XN-3,1)/2.)
      G(1,1) = S11(1)*SNS + S21(1)*CSS
      G(2,1) = SNS*S12(1) + CSS/E22(1) + S12S**((XN-1,1)/2.)*CSS/E22(1)
     1 + CSS *(XN-1,1)*S12S**((XN-3,1)/2.)/E22(1)*TS22S
     2 + (XN-1,1)*C12S*SN2*S12S**((XN-3,1)/2.)/2.*G12(1)
      G(3,1) = SN2/(2.*G12(1)) + SN2*S12S**((XN-1,1)/2.)/2.*G12(1)
     1 + SN2*(XN-1,1)/2.*G12(1)*S12S**((XN-3,1)/2.)*T12S
     2 + (XN-1,1)*C12T*S12S**((XN-3,1)/2.)*CSS/E22(1)
      H(1,1) = (S11(1)-S21(1))*SN2/2.
      H(2,1) = S12(1)*SN2/2. - SN2/(2.*E22(1)) - S12S**((XN-1,1)/2.)*SN2
     1 /2.*E22(1) - (XN-1,1)*S12S**((XN-3,1)/2.)*SN2/(2.*E22(1))
     2 + TS22S *(XN-1,1)*C12S*S12S**((XN-3,1)/2.)
     3 *CS2/(2.*G12(1))
      H(3,1) = CS2/(2.*G12(1)) + S12S**((XN-1,1)/2.)*CS2/(2.*G12(1))
     1 + (XN-1,1)*S12S**((XN-3,1)/2.)*T12S*CS2/(2.*G12(1))
     2 - (XN-1,1)*C12T*S12S**((XN-3,1)/2.)*SN2/(2.*E22(1))
      C
      RETURN
      END

```

```

0001      SUBROUTINE PESEI(I13,SG,RVAL)
C
0002      IMPLICIT      REAL*8 (A-L,O-Z)
0003      DIMENSION      SG(20,1)
C
0004      SGMX = DABS(SG(1,1))
0005      DO 318 K=2,I13
0006      IF(CABS(SG(K,1)).GT.SGMX) SGMX = DABS(SG(K,1))
0007      318 CONTINUE
0008      DO 319 K=1,20
0009      RAT = CABS(SG(K,1))/SGMX
0010      IF(RAT.LT.RVAL)      SG(K,1) = 0.00 00
0011      319 CONTINUE
0012      RETURN
0013      END

```

```

0001 SUBROUTINE INPUT2(E11,E22,V12,V12,V21,G12,SY,TY,I)
      C
      C
      C
0002 IMPLICIT REAL*8(A-Z)
0003 DOUBLE PRECISION DSORT
0004 DIMENSION E11(20),E22(20),V12(20),V21(20),G12(20),SY(20),
      GSA(11),SG12(11),GAM(20),TAU(20),TY(20)
0005 I,J,NUMT,NUMS,12
      C
0006 READ(5,1002) EF,MUF,GF,VF
0007 READ(5,1002) EM,MUM,GM
0008 READ(5,1005) I2
0009 VM = 1.0D 0 - VF
0010 TRM1 = (1.0D0+VF)/(1.0D0-VF) * GM
0011 WRITE(6,1567) I,EF,MUF,GF,VF,EM,MUM,GM,VM
0012 IF(12.EC.1) GO TO 30
0013 READ(5,1002) SY(I),TY(I)
0014 G12(I) = TRM1
0015 GO TO 60
0016 30 CONTINUE
0017 READ(5,1002) TY(I)
0018 READ(5,1005) NUMT
0019 READ(5,1002) (TAU(J),J=1,NUMT)
0020 READ(5,1002) (GAM(J),J=1,NUMT)
0021 READ(5,1002) (SG12(J),J=2,11)
      C
      C
      C
0022 SUMS6 = 0.0D0
0023 SUMS4 = 0.0D0
0024 SUMS3 = 0.0D0
0025 DO 40 J=1,NUMT
0026 SUMS6 = SUMS6 + TAU(J)**6
0027 SUMS4 = SUMS4 + TAU(J)**4
0028 SUMS3 = SUMS3 + GAM(J)*TAU(J)**3
0029 40 CONTINUE
0030 TAU = DSORT(SUMS6/(GM*SUMS3-SUMS4))
      C
      C
0031 COMPUTE SY AND G12
0032 SG12(1) = 0.0D0
0033 SUMS6 = 0.0D0
0034 SUMS4 = 0.0D0
0035 TRM2 = (3.0D0+13.0D0*VF)/(3.0D0*(1.0D0+VF)**3) + VF**2/
      (3.0D0*(1.0D0+VF)**3) + (VF/(1.0D0+VF))**3/3.0D0
      C
0036 DO 50 J=1,11
0037 GSA(J) = TRM1/(SG12(J)/TAU)**2*TRM2+1.0D0)
0038 SUMS6 = SUMS6 + SG12(J)**6
0039 SUMS4 = SUMS4 + SG12(J)**4*(GSA(1)/GSA(J)-1.0D0)
0040 50 CONTINUE
0041 SY(1) = DSORT(SUMS6/SUMS4)
      C
      C
      C
0042 60 CONTINUE
0043 PI = 3.1415900
0044 KF = EF/12.0D0*(1.0D0+MUF)*(1.0D0-2.0D0*MUF)
0045 KM = EM/12.0D0*(1.0D0+MUM)*(1.0D0-2.0D0*MUM)
0046 M = (KF*(PI+3.0D0*VF*MUM) + 2.0D0*KM*(PI-4.0D0*VF)*MUM)/
      (KF*(PI-4.0D0*VF) + 2.0D0*KM*(PI*MUM+2.0D0*VF))
0047 1 KK = KM/2.0D0*((PI*M+4.0D0-PI)/4.0D0 +

```

```

1      4.000*M/(PI*(4.000-PI)*M))
0048      E11(I) = EF*VF + FM*(1.000-VF)
0049      V12(I) = MUF*VF + MUM*(1.000-VF)
0050      MU23 = MUF*VF + MUM*VM*(1.-VUM-V12(I)*(FM/E11(I)))
0051      E22(I) = 2.0D 0*KK*E11(I)*(1.000-MU23)/(E11(I)+4.000*KK*
1      V12(I)**2)
0052      V21(I) = V12(I)*E22(I)/E11(I)
0053      1002 FORMAT (5C15.5)
0054      1003 FORMAT (5I5)
0055      1567 FORMAT (14,2(7X, 4(D11.4,2X)))
C
0056      RETURN
0057      END

```



R12D0001

SUBROUTINE INVRTD(A,NDIM,N,DETA,EPS,FRANK)

0001 C  
C

```
0002 DIMENSION A(NDIM,NDIM) R12D0002
0003 DOUBLE PRECISION PIV2,A,DETA,TEST,X,PIV,PIV1,TCL,TEMP,SUM,EPS, R12D0003
      1 DMA,DSQRT,DABS,CBLE,EPS R12D0004
0004 INTEGER IR(50),IC(50),P,S R12D0005
0005 CFTA=1. R12D0006
0006 SUM=0. R12D0007
0007 DO 10 I = 1,N R12D0008
0008   DO 10 J = 1,N R12D0009
0009     SUM=SUM+A(I,J)**2 R12D0010
0010     SUM=DSQRT(SUM) R12D0011
0011     DMA = N**2 R12D0012
0012     RMS=SUM/DMA R12D0013
0013     TOL=EPS*RMS R12D0014
0014     DO 20 I = 1,N R12D0015
0015       IP(I)=0 R12D0016
0016       DO 20 IC(I)=0 R12D0017
0017       S=0 R12D0018
0018       R = N R12D0019
0019       DO 30 I=0 R12D0020
0020         J=0 R12D0021
0021         TEST=0.0 R12D0022
0022         DO 50 K = 1,N R12D0023
0023           IF(IR(K).NE.0)GO T050 R12D0024
0024           DO 40 L = 1,N R12D0025
0025             IF(IC(L).NE.0)GO T040 R12D0026
0026             X=DABS(A(K,L)) R12D0027
0027             IF(X.LT.TEST)GO T040 R12D0028
0028             I=K R12D0029
0029             J=L R12D0030
0030             TEST=X R12D0031
0031             40 CONTINUE R12D0032
0032             50 CONTINUE R12D0033
0033             PIV=A(I,J) R12D0034
0034             IF(DETA.LT.1.0D-73) DETA = 0.9D 00 P12D0035
0035             DETA=PIV*DETA R12D0036
0036             IF (DABS(PIV) .LE. TCL) GO TC 150 R12D0037
0037             IR(I)=J R12D0038
0038             IC(J)=I R12D0039
0039             PIV = 1.0D0/PIV R12D0040
0040             ALL,JJ=PIV R12D0041
0041             DO 60 K = 1,N R12D0042
0042             60 IF(K.NE.JJA(I,K)=A(I,K)*PIV R12D0043
0043             DO 90 K = 1,N R12D0044
0044             IF (K.EQ.J) GO TO 90 R12D0045
0045             PIV1 = A(K,J) R12D0046
0046             DO 80 L = 1,N R12D0047
0047             80 IF(L.NE.JJA(K,L)=A(K,L)-PIV1*A(I,L) R12D0048
0048             90 CONTINUE R12D0049
0049             DO 100 K = 1,N R12D0050
0050             100 IF(K.NE.JJA(K,J)=-PIV*A(K,J) R12D0051
0051             S=S+1 R12D0052
0052             IF(S.LT.PICC TC 30 R12D0053
0053             110 DO 140 I = 1,N R12D0054
0054             K=IC(I) R12D0055
0055             S=I-(I) R12D0056
0056             IF(K.EQ.1)GO TC 140 R12D0057
0057             DETA=-DETA R12D0058
0058             DO 120 L = 1,N
```

0059	TEMP=A(K,I)	R1200059
0060	A(K,I)=A(I,I)	R1200060
0061	120 A(I,I)=TEMP	R1200061
0062	DO 130 I = 1,N	R1200062
0063	TEMP=A(I,M)	R1200063
0064	A(I,M)=A(I,I)	R1200064
0065	130 A(I,I)=TEMP	R1200065
0066	IC(M)=K	R1200066
0067	IR(K)=M	R1200067
0068	140 CONTINUE	R1200068
0069	150 IRANKS	R1200069
0070	RETURN	R1200070
0071	END	

### SAMPLE PROBLEM

LAYER	THETA	T	E11	E22	V12	V21	G12	SIG Y	TAU Y
1	-45	0.333300 00	0.287550 08	0.218550 07	0.255000 00	0.193810-01	0.987000 06	0.100000 05	0.500000 04
2	0	0.333300 00	0.287550 08	0.218550 07	0.255000 00	0.193810-01	0.987000 06	0.100000 05	0.500000 04
3	45	0.333300 00	0.287550 08	0.218550 07	0.255000 00	0.193810-01	0.987000 06	0.100000 05	0.500000 04

EXPCNENT M = C.30000D 01  
EXPCNENT N = C.30000D 01

# EXTERNAL APPLIED LOADING

SOLUTION FCR STRESS CONVERGES WITHIN 2 ITERATIONS

STRAIN  
(LAMINATE AXES)

LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.116590 05	C.900570 03	0.689300 03	0.414730-03	-0.302590-03	0.470920-03	-0.414730-03	0.526990-03	0.358660-03
2	-0.118190 05	-0.418000 03	0.898960 03	0.414730-03	-0.302590-03	0.470920-03	0.414730-03	-0.302550-03	0.470920-03
3	G.150000 05	-0.802210 03	-0.692220 03	0.414730-03	-0.302590-03	0.470920-03	0.526990-03	-0.414850-03	-0.358660-03

$$\begin{array}{r} F_X = 0.10000005 \\ F_Y = 0.00000000 \\ F_{XY} = 0.10000005 \end{array}$$

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRAIN  
(LAMINATE AXES)

LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY
121						



1 -0.23760 05 0.160510 04 0.134380 04 0.838070-03 -0.622150-03 0.548300-03 0.840040-03 0.105660-02 0.730410-03  
 2 0.239110 03 -0.801350 03 0.167360 04 0.638050-03 -0.622150-03 0.948290-03 0.638050-03 -0.622150-03 0.948290-03  
 3 0.300800 03 -0.115560 04 -0.132590 04 0.638040-03 -0.622150-03 0.548300-03 0.105650-02 -0.840060-03 -0.730390-03

## EXTERNAL APPLIED LOADING

F X = 0.150000 05  
 F Y = 0.000000 00  
 F XY = 0.150000 05

## SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS X	EPS Y	EPS XY
1	-0.361640 05	0.232490 04	0.185490 04	0.127480-02	-0.964720-03	-0.127830-02	0.158840-02	0.111980-02
2	0.363660 05	-0.114470 04	0.230770 04	0.127480-02	-0.964720-03	0.127480-02	-0.964720-03	0.143340-02
3	0.452570 05	-0.163850 04	-0.168580 04	0.127480-02	-0.964720-03	0.158840-02	-0.127830-02	-0.111980-02

## EXTERNAL APPLIED LOADING

F X = 0.200000 05  
 F Y = 0.000000 00  
 F XY = 0.200000 05

## SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS X	EPS Y	EPS XY
1	-0.469630 05	0.284390 04	0.232240 04	0.172220-02	-0.132680-02	-0.172800-02	0.212140-02	0.152550-02
2	0.491510 05	-0.145430 04	0.283140 04	0.172220-02	-0.132680-02	0.172220-02	-0.132680-02	0.152470-02
3	0.604770 05	-0.205420 04	-0.237510 04	0.172220-02	-0.132680-02	0.212140-02	-0.172800-02	-0.152550-02

## EXTERNAL APPLIED LOADING

F X = 0.250000 05  
 F Y = 0.000000 00  
 F XY = 0.250000 05

## SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

Y

## STRESS

STRAIN  
(LAMINATE AXES)STRAIN  
(LAYER AXES)

LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.620400 05	0.324250 04	0.273130 04	0.217870-02	-0.171070-02	0.242070-02	-0.218670-02	0.269470-02	0.194470-02
2	0.622040 05	-0.173910 04	0.321780 04	0.217860-02	-0.171070-02	0.242060-02	0.217860-02	-0.171070-02	0.242060-02
3	0.757190 05	-0.241450 04	-0.279670 04	0.217850-02	-0.171070-02	0.242070-02	0.269470-02	-0.218680-02	-0.194460-02

## EXTERNAL APPLIED LOADING

F X = 0.300000 05  
F Y = 0.000000 00  
F XY = 0.300000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

## STRESS

STRAIN  
(LAMINATE AXES)STRAIN  
(LAYER AXES)

LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.759480 05	0.363470 04	0.309410 04	0.264240-02	-0.210710-02	0.292020-02	-0.265260-02	0.318790-02	0.237470-02
2	0.754710 05	-0.199470 04	0.365900 04	0.264230-02	-0.210710-02	0.292010-02	0.264230-02	-0.210710-02	0.292010-02
3	0.902680 05	-0.273090 04	-0.311730 04	0.264220-02	-0.210710-02	0.292020-02	0.318780-02	-0.265270-02	-0.237460-02

## EXTERNAL APPLIED LOADING

F X = 0.350000 05  
F Y = 0.000000 00  
F XY = 0.350000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

## STRESS

STRAIN  
(LAMINATE AXES)STRAIN  
(LAYER AXES)

LAYER	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.866260 05	0.394760 04	0.342030 04	0.311180-02	-0.251520-02	0.362240-02	-0.312410-02	0.372070-02	0.281350-02
2	0.889090 05	-0.243380 04	0.399660 04	0.311180-02	-0.251520-02	0.342220-02	0.311180-02	-0.251520-02	0.342220-02
3	0.106220 06	-0.301280 04	-0.330820 04	0.311160-02	-0.251520-02	0.342240-02	0.372060-02	-0.312420-02	-0.281340-02

## EXTERNAL APPLIED LOADING

F X = 0.400000 05  
F Y = 0.000000 00  
F XY = 0.400000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS

LAYER	STRESS (LAMINATE AXES)			STRAIN (LAYER AXES)		
	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY
1	-0.102440 06	0.422280 04	0.371710 04	0.358550-02	-0.293300-02	0.392660-02
2	0.102490 06	-0.245920 04	0.430330 04	0.358550-02	-0.293300-02	0.392660-02
3	0.121460 06	-0.326700 04	-0.361180 04	0.358570-02	-0.293300-02	0.392660-02

EXTERNAL APPLIED LOADING

F X = 0.400000 05  
F Y = 0.000000 00  
F XY = 0.400000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS

LAYER	STRESS (LAMINATE AXES)			STRAIN (LAYER AXES)		
	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY
1	-0.116180 06	0.446860 04	0.396560 04	0.406390-02	-0.335890-02	0.443240-02
2	0.116180 06	-0.256430 04	0.456030 04	0.406390-02	-0.335890-02	0.443240-02
3	0.136690 06	-0.349670 04	-0.406960 04	0.406360-02	-0.335890-02	0.443240-02

EXTERNAL APPLIED LOADING

F X = 0.500000 05  
F Y = 0.000000 00  
F XY = 0.500000 05

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS

LAYER	STRESS (LAMINATE AXES)			STRAIN (LAYER AXES)		
	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY
1	-0.130010 06	0.469090 04	0.424160 04	0.454510-02	-0.379150-02	0.493560-02
2	0.129560 06	-0.265960 04	0.463460 04	0.454510-02	-0.379150-02	0.493560-02
3	0.151920 06	-0.371200 04	-0.434590 04	0.454510-02	-0.379150-02	0.493560-02

## EXTERNAL APPLIED LOADING

F X = 0.550000 05  
 F Y = 0.000000 00  
 F XY = 0.550000 05

## SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

LAYER	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.145910 06	0.439420 04	0.447690 04	0.502500-02	-0.422590-02	0.544780-02	-0.506830-02	0.586740-02	0.462950-02
2	0.143850 06	-0.304530 04	0.507010 04	0.502500-02	-0.422590-02	0.544750-02	0.502500-02	-0.422590-02	0.544750-02
3	0.167140 06	-0.390580 04	-0.456420 04	0.502870-02	-0.422590-02	0.544780-02	0.586730-02	-0.504840-02	-0.462530-02

## EXTERNAL APPLIED LOADING

F X = 0.600000 05  
 F Y = 0.000000 00  
 F XY = 0.600000 05

## SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

LAYER	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.157890 06	0.508170 04	0.469730 04	0.551530-02	-0.467330-02	0.595700-02	-0.553600-02	0.637800-02	0.509430-02
2	0.157770 06	-0.321910 04	0.526980 04	0.551530-02	-0.467330-02	0.595670-02	0.551530-02	-0.467330-02	0.555670-02
3	0.182350 06	-0.405440 04	-0.480700 04	0.551530-02	-0.467330-02	0.595700-02	0.637790-02	-0.553620-02	-0.509420-02

## EXTERNAL APPLIED LOADING

F X = 0.650000 05  
 F Y = 0.000000 00  
 F XY = 0.650000 05

## SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

LAYER	STRESS			STRAIN (LAMINATE AXES)			STRAIN (LAYER AXES)		
	SGM X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY

LAYER	SGP X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY
1	-0.17183D 06	0.22259D 04	1.49049D 04	0.60036D-02	-0.51211D-02	0.64670D-02	-0.60257D-02	0.69083D-02	0.55624D-02
2	0.17177D 06	-0.33833D 04	0.54938D 04	0.60037D-02	-0.51211D-02	0.64667D-02	0.60037D-02	-0.51211D-02	0.64667D-02
3	0.19756D 06	-0.42678D 04	-0.50163D 04	0.60034D-02	-0.51211D-02	0.64670D-02	0.69081D-02	-0.60259D-02	-0.55622D-02

LAMINATE HAS FAILED; SGM X EXCEEDS MAXIMUM

AT FAILURE  
EXTERNAL APPLIED LOADING

F X = 0.70000D 09  
F Y = 0.00000D 00  
F XY = 0.70000D 09

SOLUTION FOR STRESS CONVERGES WITHIN 2 ITERATIONS

STRESS				STRAIN (LAMINATE AXES)				STRAIN (LAYER AXES)			
LAYER	SGP X	SGM Y	SGM XY	EPS X	EPS Y	EPS XY	EPS X	EPS Y	EPS XY	EPS X	EPS XY
1	-0.18062D 06	0.54169D 04	0.51014D 04	0.64937D-02	-0.55728D-02	0.69777D-02	-0.65172D-02	0.71562D-02	0.60332D-02	-0.65172D-02	0.71562D-02
2	0.18062D 06	-0.55450D 04	0.50900D 04	0.64938D-02	-0.55728D-02	0.69777D-02	0.64938D-02	-0.55728D-02	0.60332D-02	0.64938D-02	-0.55728D-02
3	0.21275E 06	-0.44311D 04	-0.52144D 04	0.64935D-02	-0.55728D-02	0.69777D-02	0.74350D-02	-0.65173D-02	-0.60333D-02	0.74350D-02	-0.65173D-02

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